



## Behavior of the standard Dickey–Fuller test when there is a Fourier-form break under the null hypothesis<sup>☆</sup>



Lixiong Yang <sup>a,\*</sup>, Chingnun Lee <sup>b</sup>, Jen-Je Su <sup>c</sup>

<sup>a</sup> School of Management, Lanzhou University, Lanzhou, China

<sup>b</sup> Institute of Economics, National Sun Yat-sen University, Kaohsiung, Taiwan

<sup>c</sup> Department of Accounting Finance and Economics, Griffith University, Brisbane, Australia

### HIGHLIGHTS

- We assume that the null unit root process is with a Fourier component.
- We derive the asymptotic distribution of the standard Dickey–Fuller (DF) test.
- Asymptotic distributional results generate interesting predictions.
- The converse Perron phenomenon may occur when a Fourier-form break exists.
- The predictions are confirmed by simulation results.

### ARTICLE INFO

*Article history:*

Received 25 February 2017

Received in revised form 21 June 2017

Accepted 14 July 2017

Available online 29 July 2017

### ABSTRACT

We derive the null asymptotic distribution of the standard Dickey–Fuller test with the existence of an unnoticed Fourier component. The so-called converse Perron phenomenon might occur, but only in the trend-case with a low-frequency Fourier component and small error variance.

© 2017 Elsevier B.V. All rights reserved.

*JEL classification:*

C12

C15

*Keywords:*

Structural break

Fourier approximation

Unit root test

## 1. Introduction

Following the seminal work of Perron (1989), it is now a well-known fact that the usual Dickey–Fuller (DF) test is inconsistent when applied to stationary series with a break. In contrast, Leybourne et al. (1998) and Leybourne and Newbold (2000) illustrate a “converse Perron phenomenon”, suggesting that the usual DF test tends to suffer enormous size distortion when applied to a unit root process with an abrupt break (particularly, if the break occurs early in the sample). Accordingly, the usual DF test is likely to mix up a stationary series carrying a break with a unit root process and mistake a unit root process with a break for a stationary series.

Since 1989, a vast literature has developed around incorporating breaks into unit root testing. The literature begins with considering a single exogenous break (i.e. a break at a known point) and steadily evolves into permitting for possible multiple endogenous breaks (i.e. breaks at unknown dates); see Perron (2006) for a comprehensive survey. In practice, this line of research requires to assume the maximum number of breaks and identify the break dates. These parameters are crucial to the performance of break-adjusted unit root tests but they are hard to be properly estimated.

Becker et al. (2006) and Enders and Lee (2012a,b) suggest a new approach in handling breaks for unit root tests. They demonstrate that the flexible Fourier expansion of Gallant (1981) can well approximate the deterministic component of an economic time series with numerous breaks. The new approach is advantageous for its simplicity as commonly only a single frequency is sufficient to achieve a reasonable approximation. In terms of empirical relevance, according to Enders and Lee (2012a,b), using the specific

<sup>☆</sup> We thank an anonymous referee and the editor for very valuable comments and suggestions. Remaining errors are our own.

\* Correspondence to: School of Management, Lanzhou University, 222 South Tianshui Road, Lanzhou, Gansu, 730000, China

E-mail address: [ylx@lzu.edu.cn](mailto:ylx@lzu.edu.cn) (L. Yang).

frequency  $k = 1$  often leads to a good approximation for breaks of unknown form in economic series.

According to Enders and Lee (2012a), ignoring Fourier-type breaks in a stationary series can lead to a Perron-like phenomenon of inconsistency. However, the literature is silent if a similar converse Perron phenomenon will occur when a unit root process with a Fourier component is considered. The main purpose of this paper is to fill this gap in the literature. To this end, we derive the asymptotic distribution of the DF  $t$ -statistic under the null hypothesis, assuming the null unit root process is accompanied with a Fourier component. Interestingly, we find that ignoring a Fourier component will end up with very different outcomes: the null hypothesis can be either over-rejected or under-rejected, depending on the setting of the Fourier component, the variance of the disturbance, and whether the DF test allows for a linear trend. In other words, the converse Perron phenomenon can arise, but only in certain cases. All the results in this note are derived by assuming the integer frequency  $k$  and “ $\Rightarrow$ ” stands for weak convergence.

## 2. The standard DF test under a Fourier-form break

Let  $y_t$  be generated by the following AR(1) model

$$(1 - \phi L)(y_t - \alpha(t) - \gamma t) = u_t, \quad t = 1, 2, \dots, T, \quad (1)$$

where  $u_t$  is an i.i.d. disturbance with zero mean and constant variance  $\sigma^2$ ,  $\alpha(t)$  is a time-varying deterministic break function, and  $\gamma t$  is a linear deterministic trend. The initial value  $y_0$  is assumed to be  $O(1)$ . Following Enders and Lee (2012a, b) and Lee et al. (2016),  $\alpha(t)$  is set to the following single-frequency Fourier form:<sup>1</sup>

$$\alpha(t) = \alpha_0 + \beta_1 \sin(2\pi kt/T) + \beta_2 \cos(2\pi kt/T), \quad (2)$$

where  $\beta_1$  and  $\beta_2$  measure the amplitude and displacement of sinusoidal components and  $k$  represents a particular frequency.

We are of interest to test for a unit root ( $\phi = 1$ ) against stationarity ( $\phi < 1$ ) from the standard DF test. Specifically, we aim to examine the situation under the unit root null hypothesis when the Fourier component ( $\alpha(t)$ ) in (1) is unnoticed. Similar to Leybourne et al. (1998) and Leybourne and Newbold (2000), we assume the magnitude of the break amplitude parameters,  $\beta$ 's, is proportional to  $T^{1/2}$ . This assumption ensures that, asymptotically, the break component  $\alpha(t)$  and the random walk component of  $y_t$  are of the same order of magnitude (in probability). Thus, the asymptotic distribution of the test statistics depends on the break.

**Theorem 1.** Suppose  $y_t$  is generated based on (1) and (2) with  $\phi = 1$  and assume  $\beta_1 = \kappa_1 T^{1/2}$  and  $\beta_2 = \kappa_2 T^{1/2}$ , where  $\kappa_1$  and  $\kappa_2$  are constants. We consider the following two DF tests.

(a). **Trend case:** Let  $t^{DF_{t,B}}$  be the standard DF  $t$ -test statistics of the regression:  $\Delta y_t = \rho y_{t-1} + c_1 + c_2 t + e_t$ . We have<sup>2</sup> Eq. (3) is given in Box I, in which

$$\begin{aligned} \mathbf{q} &= \begin{bmatrix} W(1) \\ W(1) - \int_0^1 W(r)dr \end{bmatrix}, & \mathbf{h} &= \begin{bmatrix} \int_0^1 W(r)dr \\ \int_0^1 rW(r)dr \end{bmatrix}, \\ \Psi &= \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}, \end{aligned}$$

<sup>1</sup> Without loss of generality, it is assumed that  $\alpha(0) = 0$ .

<sup>2</sup> In Appendix A, we have shown that (from (A.7)), for large  $\kappa$ 's, the bias of the estimator of  $\sigma^2$  is of the same order of magnitude (in probability) with the term,  $\frac{1}{2} \left\{ \frac{(2\pi k)^2 \kappa_1^2}{T} + \frac{(2\pi k)^2 \kappa_2^2}{T} \right\}$ . Hence, for a small  $\sigma^2$  and large  $\kappa$ 's, the bias is considerable in finite sample size. It therefore needs a large sample size  $T$  to obtain this asymptotic result under this circumstance.

$$\begin{aligned} d_1 &= \int_0^1 W(r)dr + 2\pi k \int_0^1 \sin(2\pi kr) \left[ \int_0^r W(s)ds \right] dr, \\ d_2 &= -2\pi k \left( \int_0^1 \cos(2\pi kr) \left[ \int_0^r W(s)ds \right] dr \right) \\ &\quad - \frac{3}{\pi k} \left( \int_0^1 W(r)dr - 2 \int_0^1 rW(r)dr \right), \\ d_3 &= -2\pi k \int_0^1 \cos(2\pi kr) W(r)dr - \frac{3}{\pi k} \left[ 2 \int_0^1 W(r)dr - 3W(1) \right], \\ d_4 &= W(1) + 2\pi k \int_0^1 \sin(2\pi kr) W(r)dr. \end{aligned}$$

(b). **Mean case:** If  $\gamma = 0$  in (1), and let  $t^{DF_{c,B}}$  be the standard DF  $t$ -test statistics of the regression:  $\Delta y_t = \rho y_{t-1} + c_1 + e_t$ , then Eq. (4) is given in Box II, in which  $d_2^* = -2\pi k \int_0^1 \cos(2\pi kr) \left[ \int_0^r W(s)ds \right] dr$  and  $d_3^* = -2\pi k \int_0^1 \cos(2\pi kr) W(r)dr$ .

**Proof.** See Appendix A. ■

Clearly, the asymptotic distribution of the DF statistic depends on  $\sigma$ ,  $k$ ,  $\kappa_1$  and  $\kappa_2$  except for the special case of no breaks ( $\kappa_1 = \kappa_2 = 0$ ).<sup>3</sup> Hence, ignoring the Fourier component may lead to non-trivial size distortion. However, as the asymptotic distribution is complicated, it is hard to quantify the extent of inconsistency. To focus on the impact of the size when the Fourier components are ignored, following Leybourne and Newbold (2000), further insight can be shed by considering the extreme case of large  $\kappa$ 's. For large  $\kappa_1$  and  $\kappa_2$ , as  $T \rightarrow \infty$ , then approximately<sup>4</sup>

$$t^{DF_{t,B}} \rightarrow \frac{\frac{6\kappa_1\kappa_2}{\pi k\sigma}}{\left( \frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{3\kappa_1^2}{(\pi k)^2} \right)^{1/2}}, \quad (\text{Trend}) \quad (5)$$

and<sup>5</sup>

$$t^{DF_{c,B}} \rightarrow 0. \quad (\text{Mean}) \quad (6)$$

The result of (6) implies that, in the mean case, as  $t^{DF_{c,B}} \rightarrow 0$ , the DF test is likely to under-reject the null hypothesis when  $\kappa$ 's are large. On the other hand, for the trend case, since the limiting distribution of  $t^{DF_{t,B}}$  in (5) hinges on  $k$ ,  $\kappa_1$ ,  $\kappa_2$  and  $\sigma$ , the direction of inconsistency is unclear. We plot the limiting distribution in (5) with two illustrative examples: Fig. 1(a) for  $\kappa_1 = 2$  and  $\kappa_2 = -2$  and Fig. 1(b) for  $\kappa_1 = \kappa_2 = 2$ , under various combinations of  $\sigma$  and  $k$ . Fig. 1(a) shows that, when  $\kappa_1$  and  $\kappa_2$  are large with opposite signs and  $\kappa \cdot \sigma$  is small,  $t^{DF_{t,B}}$  converges to a low negative value and the null hypothesis is likely to be over-rejected. However, the chance of over-rejection lessens as  $k \cdot \sigma$  increases. Conversely, according to Fig. 1(b), when  $\kappa_1$  and  $\kappa_2$  are with the same sign, since  $t^{DF_{t,B}}$  converges to a positive value, under-rejection is likely to occur for any combination of  $\sigma$  and  $k$ .

## 3. Simulation evidence

In this section, we examine the performance of the DF test using

<sup>3</sup> When  $\kappa_1 = \kappa_2 = 0$ , the distribution shrinks to the usual DF distribution. For example,  $t^{DF_{t,B}} \rightarrow \frac{\int_0^1 W(r)dW(r) - \mathbf{q}' \Psi^{-1} \mathbf{h}}{\int_0^1 W^2(r)dr - \mathbf{h}' \Psi^{-1} \mathbf{h}}$ .

<sup>4</sup> Simulation results show that  $d_1$ ,  $d_2$ ,  $d_3$  and  $d_4$  in (3) are largely symmetric around zero and mostly lie between  $-0.1$  and  $0.1$ . Therefore, they are negligible when  $\kappa$ 's are large.

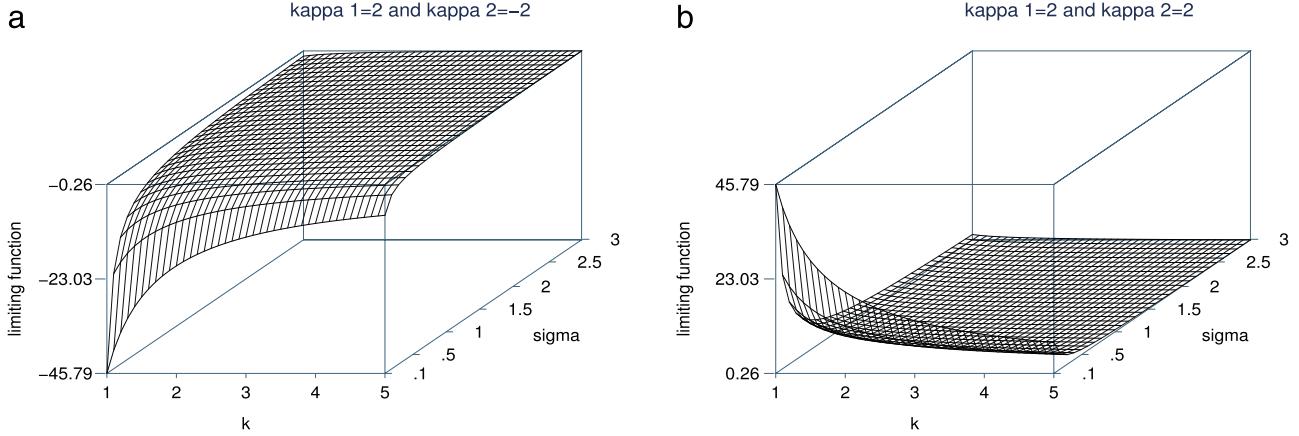
<sup>5</sup> When  $\kappa$ 's are large,  $\frac{1}{2}(\kappa_1^2 + \kappa_2^2)$  in the denominator of (4) dominates all other terms and, as a result,  $t^{DF_{c,B}}$  converges to zero. Simulation results also show that  $d_2^*$  and  $d_3^*$  are symmetric around zero and, for most cases, less than  $10^{-5}$  in magnitude.

$$t^{DF_{t,B}} \Rightarrow \frac{\int_0^1 W(r)dW(r) - \mathbf{q}'\Psi^{-1}\mathbf{h} + 2\pi k\kappa_1\mathbf{d}_1 - 2\pi k\kappa_2\mathbf{d}_2 + \kappa_1\mathbf{d}_3 + \kappa_2\mathbf{d}_4 + \frac{6\kappa_1\kappa_2}{\pi k\sigma}}{\left(\int_0^1 W^2(r)dr - \mathbf{h}'\Psi^{-1}\mathbf{h} + 2\sigma\kappa_1\mathbf{d}_2 + 2\sigma\kappa_2\mathbf{d}_1 + \frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{3\kappa_1^2}{(\pi k)^2}\right)^{1/2}}, \quad (3)$$

Box I.

$$t^{DF_{c,B}} \Rightarrow \frac{\frac{1}{2}(W^2(1) - 1) - W(1)\int_0^1 W(r)dr + 2\pi k\kappa_1\mathbf{d}_1 - 2\pi k\kappa_2\mathbf{d}_2^* + \kappa_1\mathbf{d}_3^* + \kappa_2\mathbf{d}_4}{\left(\int_0^1 W^2(r)dr - \left[\int_0^1 W(r)dr\right]^2 + 2\sigma\kappa_1\mathbf{d}_2^* + 2\sigma\kappa_2\mathbf{d}_1 + \frac{1}{2}(\kappa_1^2 + \kappa_2^2)\right)^{1/2}}, \quad (4)$$

Box II.



**Fig. 1.** Limiting function of DF in (5) for large amplitude parameters: (a)  $\kappa_1 = 2$  and  $\kappa_2 = -2$ ; (b)  $\kappa_1 = 2$  and  $\kappa_2 = 2$ .

**Table 1**  
Size of standard DF test with trend when DGP is given as (1) and (2). (Under a fixed  $\sigma = 1$ ).

T	$(\kappa_1, \kappa_2)$	(0.1, -0.1)	(0.1, 0.1)	(0.5, -0.5)	(0.5, 0.5)	(1, -1)	(1, 1)	(1.5, -1.5)	(1.5, 1.5)	(2, -2)	(2, 2)
100	$k = 1$	0.043	0.046	0.024	0.001	0.118	0.000	0.439	0.000	0.802	0.000
	$k = 2$	0.048	0.055	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.052	0.056	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.045	0.043	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.041	0.043	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
200	$k = 1$	0.052	0.049	0.022	0.001	0.142	0.000	0.577	0.000	0.936	0.000
	$k = 2$	0.055	0.054	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.050	0.055	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.046	0.047	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.042	0.043	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
500	$k = 1$	0.052	0.047	0.013	0.003	0.151	0.000	0.673	0.000	0.979	0.000
	$k = 2$	0.051	0.052	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.049	0.051	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.045	0.048	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.042	0.042	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Notes:  $y_t$  is generated as  $(1 - \phi)L(y_t - \alpha(t) - \gamma t) = u_t$ ,  $\alpha(t) = \alpha_0 + \beta_1 \sin(2\pi kt/T) + \beta_2 \cos(2\pi kt/T)$ , with  $\phi = 1$ ,  $\beta_1 = \kappa_1\sqrt{T}$ ,  $\beta_2 = \kappa_2\sqrt{T}$ ,  $\gamma = 1$ ,  $\alpha_0 = 1$  and  $u_t \stackrel{i.i.d.}{\sim} N(0, \sigma^2)$  with  $\sigma = 1$ . Size is computed at the 5% nominal level, based on 2000 replications.

Monte Carlo experiments. We generate the time series through the data generating process given as (1) and (2) with  $\phi = 1$  and  $u_t$  being drawn from  $i.i.d. N(0, \sigma^2)$ .<sup>6</sup> We set  $T = 100, 200, 500$  and  $k = 1, 2, 3, 4, 5$ . All simulations are done with GAUSS, using 2000 replications at the 5% significance level.

In the first experiment, we choose  $(\kappa_1, \kappa_2) = \{(0.1, -0.1), (0.1, 0.1)\}, \{(0.5, -0.5), (0.5, 0.5)\}, \{(1, -1), (1, 1)\}, \{(1.5, -1.5),$

$(1.5, 1.5)\}, \{(2, -2), (2, 2)\}$ , given  $\sigma = 1$ . The results are reported in Table 1 (trend) and Table 2 (mean). The results of Table 1 can be summarized as follows. First, when  $\kappa_1$  and  $\kappa_2$  are small in magnitude (e.g.,  $\pm 0.1$ ), the DF test is largely correct-sized. Second, when the Fourier frequency  $k$  is 2 or larger, the DF test tends to under-reject the null hypothesis when  $|\kappa_1|$  and  $|\kappa_2|$  are moderate (i.e.,  $|\kappa_1| = |\kappa_2| = 0.5$ ) or large (i.e.,  $|\kappa_1| \geq 1$  and  $|\kappa_2| \geq 1$ ). Third, for  $k = 1$ , over-rejection emerges when  $|\kappa_1| \geq 1$  and  $|\kappa_2| \geq 1$ , and  $\kappa_1$  and  $\kappa_2$  are with different signs. The rate of over-rejection

<sup>6</sup> Since the DF test in the trend case is free from  $\gamma$  and  $\alpha_0$ , without loss of generality, we set  $\gamma = \alpha_0 = 1$ .

**Table 2**

Size of standard trend-free DF test when DGP is given as (1) and (2) with  $\gamma = 0$ . (Under a fixed  $\sigma = 1$ ).

T	$(\kappa_1, \kappa_2)$	(0.1, -0.1)	(0.1, 0.1)	(0.5, -0.5)	(0.5, 0.5)	(1, -1)	(1, 1)	(1.5, -1.5)	(1.5, 1.5)	(2, -2)	(2, 2)
100	$k = 1$	0.053	0.043	0.028	0.003	0.022	0.000	0.001	0.000	0.001	0.000
	$k = 2$	0.048	0.037	0.011	0.000	0.001	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.039	0.040	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.032	0.038	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.031	0.036	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
200	$k = 1$	0.049	0.048	0.034	0.002	0.023	0.000	0.008	0.000	0.003	0.000
	$k = 2$	0.041	0.031	0.009	0.000	0.001	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.038	0.029	0.009	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.043	0.034	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.038	0.033	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
500	$k = 1$	0.052	0.043	0.049	0.002	0.021	0.000	0.018	0.001	0.008	0.001
	$k = 2$	0.045	0.028	0.017	0.001	0.007	0.000	0.002	0.000	0.001	0.000
	$k = 3$	0.048	0.030	0.007	0.001	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.032	0.028	0.005	0.002	0.000	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.033	0.032	0.005	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Notes:  $y_t$  is the same as that in Table 1 with  $\gamma = 0$ . Size is computed at the 5% nominal level, based on 2000 replications.

**Table 3**

Size of standard DF test with trend when DGP is given as (1) and (2). (Under fixed large  $\kappa$ 's,  $\kappa_1 = 1, \kappa_2 = -1$ ).

T	$(\kappa_1, \kappa_2) = (1, -1)$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 1.5$	$\sigma = 2$	$\sigma = 2.5$
100	$k = 1$	0.813	0.118	0.035	0.030	0.028
	$k = 2$	0.000	0.000	0.000	0.000	0.001
	$k = 3$	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.000	0.000	0.000	0.000	0.000
200	$k = 1$	0.936	0.142	0.031	0.019	0.022
	$k = 2$	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.000	0.000	0.000	0.000	0.000
500	$k = 1$	0.981	0.151	0.031	0.022	0.021
	$k = 2$	0.000	0.000	0.000	0.000	0.000
	$k = 3$	0.000	0.000	0.000	0.000	0.000
	$k = 4$	0.000	0.000	0.000	0.000	0.000
	$k = 5$	0.000	0.000	0.000	0.000	0.000

Notes:  $y_t$  is the same as that in Table 1 with various  $\sigma$ . Size is computed at the 5% nominal level, based on 2000 replications.

surges as  $|\kappa_1|$  and  $|\kappa_2|$  increase.<sup>7</sup> However, if  $\kappa_1$  and  $\kappa_2$  are with the same sign, under-rejection is more likely to arise. On the other hand, Table 2 shows that, in the mean case, the DF test is correctly sized when  $|\kappa_1| = |\kappa_2| = 0.1$  and under-sized otherwise. Overall, the results in Tables 1 and 2 are in line of the asymptotic theory established in Theorem 1.

In the second experiment, we set  $\sigma = \{0.5, 1, 1.5, 2, 2.5\}$  and  $(\kappa_1, \kappa_2) = (1, -1)$ . We report the simulation results for the trend case in Table 3.<sup>8</sup> Table 3 shows that over-rejection is likely to occur only when  $k = 1$  and  $\sigma$  is relatively small (i.e.,  $\sigma \leq 1$ ) but not in other cases. The results are consistent with the asymptotic theory of Theorem 1 and Fig. 1.<sup>9</sup>

<sup>7</sup> Our unreported simulations show that the finite sample results converge to the limit size more quickly when the  $\kappa$ 's are even larger. In Appendix A, we show that  $t^{DF,B} = \frac{\frac{\sigma^2 \mathbf{v}' \mathbf{M} \mathbf{y}_{t-1}}{T} + \lambda_1^*}{\left( \frac{\sigma^2 \mathbf{v}' \mathbf{M} \mathbf{v}}{T-3} \right)^{1/2} \times \left( \frac{\sigma^2 \mathbf{s}'_{t-1} \mathbf{M} \mathbf{s}_{t-1}}{T^2} + \lambda_2^* \right)^{1/2}} \oplus O\left(\frac{1}{\sqrt{T}}\right)$ . We conjecture that the median sums of  $\oplus O\left(\frac{1}{\sqrt{T}}\right) / O\left(\frac{1}{\sqrt{T}}\right)$  may play a crucial role in the slower rate convergence at  $\kappa_1 = 2$  and  $\kappa_2 = -2$  when  $k = 1$ . We thank a referee to raise this point to us.

<sup>8</sup> The results for the mean case are available upon request.

<sup>9</sup> Fig. 1 predicts that, for any  $k$ , when  $\sigma$  is very small (say, 0.1), the DF test is likely to over-reject the null hypothesis. However, as pointed out in footnote 2, the estimation bias of  $\sigma$  is substantial when the sample size is not large enough. In fact, according to our simulation results, it will require  $T \geq 80,000$  to establish the phenomenon empirically. The results are available upon request.

#### 4. Conclusion

In this note, we examine if the converse Perron phenomenon would arise when a Fourier-form break is unspecified. We find that such misspecification will result in non-trivial size distortion and the direction and degree of distortion is subject to the setting of the Fourier component, the variance of the error term, and whether the test allows for a trend. Specifically, the converse Perron phenomenon is likely to occur when applying the DF-trend test to series with an overlooked low-frequency Fourier component and small error variance.

#### Appendix A

**Proof of Theorem 1.**<sup>10</sup> We first prove the trend case. Denote  $\mathbf{y} = (y_1, y_2, \dots, y_T)$ ,  $\boldsymbol{\gamma}_1 = (\sin(2\pi k_1 T), \sin(2\pi k_2 T), \dots, \sin(2\pi k T))'$  and  $\boldsymbol{\gamma}_2 = (\cos(2\pi k_1 T), \cos(2\pi k_2 T), \dots, \cos(2\pi k T))'$ , the matrix format of DGP ((1) and (2)) under the null hypothesis of  $\phi = 1$  is:

$$\Delta \mathbf{y} = \gamma \cdot \boldsymbol{\tau} + \beta_1 \Delta \boldsymbol{\gamma}_1 + \beta_2 \Delta \boldsymbol{\gamma}_2 + \mathbf{u}, \quad (\text{A.1})$$

where  $\boldsymbol{\tau} = (1, 1, \dots, 1)'$  and  $\mathbf{u} = (u_1, u_2, \dots, u_T)'$ . By recursively substituting (A.1), we obtain  $\mathbf{y}_{t-1}$  as:

$$\mathbf{y}_{t-1} = \mathbf{y}_0 \boldsymbol{\tau} + \gamma \mathbf{t}_{t-1} + \beta_1 \boldsymbol{\gamma}_{1,t-1} + \beta_2 \boldsymbol{\gamma}_{2,t-1} + \mathbf{s}_{t-1}, \quad (\text{A.2})$$

<sup>10</sup> The full version of the proof is available upon request.

$$t^{DF,B} \Rightarrow \frac{\int_0^1 W(r)dW(r) - \mathbf{q}'\Psi^{-1}\mathbf{h} + 2\pi k\kappa_1 d_1 - 2\pi k\kappa_2 d_2 + \kappa_1 d_3 + \kappa_2 d_4 + \frac{6\kappa_1\kappa_2}{\pi k\sigma}}{\left(\int_0^1 W^2(r)dr - \mathbf{h}'\Psi^{-1}\mathbf{h} + 2\sigma\kappa_1 d_2 + 2\sigma\kappa_2 d_1 + \frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{3\kappa_1^2}{(\pi k)^2}\right)^{1/2}}.$$

## Box III.

where  $\mathbf{t}_{-1} = (0, 1, 2, \dots, T-1)$  and  $\mathbf{s}_{y,-1} = (s_{y0}, s_{y1}, \dots, s_{y,T-1})$  with  $s_{y0} = 0$  and  $s_{yt} = u_1 + u_2 + \dots + u_t$ .

If the Fourier terms are omitted in the regression, the  $t$ -statistic in unit-root test in Dickey and Fuller (1979) is calculated as:

$$t^{DF,B} = \frac{\Delta \mathbf{y}' \mathbf{M} \mathbf{y}_{-1}}{\left(\frac{\Delta \mathbf{y}' \mathbf{M} \mathbf{y} \Delta \mathbf{y}}{T-3}\right)^{1/2} (\mathbf{y}'_{-1} \mathbf{M} \mathbf{y}_{-1})^{1/2}}, \quad (\text{A.3})$$

where  $\mathbf{M} = \mathbf{I}_T - \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$ ,  $\mathbf{Z} = (\boldsymbol{\tau}, \mathbf{t})$ ,  $\mathbf{M}_y = \mathbf{I}_T - \mathbf{G}(\mathbf{G}'\mathbf{G})^{-1}\mathbf{G}'$ , and  $\mathbf{G} = (\mathbf{Z}, \mathbf{y}_{-1})$ .

Let  $\mathbf{v} = \frac{\mathbf{u}}{\sigma}$  and  $\xi_{y,-1} = \frac{s_{y,-1}}{\sigma}$ , after tedious algebra, we have

$$t^{DF,B} = \frac{\frac{\sigma^2 \mathbf{v}' \mathbf{M} \xi_{y,-1}}{T} + \lambda_1}{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M}_y \mathbf{v}}{T-3} + \lambda_3\right)^{1/2} \times \left(\frac{\sigma^2 \xi'_{y,-1} \mathbf{M} \xi_{y,-1}}{T^2} + \lambda_2\right)^{1/2}}, \quad (\text{A.4})$$

in which

$$\begin{aligned} \lambda_1 &= \frac{1}{T} \left[ \beta_1 \sigma \frac{2\pi k}{T} \mathbf{r}'_2 \mathbf{M} \xi_{y,-1} - \beta_2 \sigma \frac{2\pi k}{T} \mathbf{r}'_1 \mathbf{M} \xi_{y,-1} + \sigma \beta_1 \mathbf{v}' \mathbf{M} \mathbf{r}_{1,-1} \right. \\ &\quad + \beta_1^2 \frac{2\pi k}{T} \mathbf{r}'_2 \mathbf{M} \mathbf{r}_{1,-1} - \beta_1 \beta_2 \frac{2\pi k}{T} \mathbf{r}'_1 \mathbf{M} \mathbf{r}_{1,-1} + \sigma \beta_2 \mathbf{v}' \mathbf{M} \mathbf{r}_{2,-1} \\ &\quad \left. + \beta_1 \beta_2 \frac{2\pi k}{T} \mathbf{r}'_2 \mathbf{M} \mathbf{r}_{2,-1} - \beta_2^2 \frac{2\pi k}{T} \mathbf{r}'_1 \mathbf{M} \mathbf{r}_{2,-1} \right] \\ &= \lambda_1^* + O\left(\frac{1}{T^2}\right), \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \lambda_2 &= \frac{1}{T^2} \left[ \sigma \beta_1 \xi'_{y,-1} \mathbf{M} \mathbf{r}_{1,-1} + \sigma \beta_2 \xi'_{y,-1} \mathbf{M} \mathbf{r}_{2,-1} + \sigma \beta_1 \mathbf{r}'_{1,-1} \mathbf{M} \xi_{y,-1} \right. \\ &\quad + \beta_1^2 \mathbf{r}'_{1,-1} \mathbf{M} \mathbf{r}_{1,-1} + \beta_1 \beta_2 \mathbf{r}'_{1,-1} \mathbf{M} \mathbf{r}_{2,-1} + \sigma \beta_2 \mathbf{r}'_{2,-1} \mathbf{M} \xi_{y,-1} \\ &\quad \left. + \beta_2 \beta_1 \mathbf{r}'_{2,-1} \mathbf{M} \mathbf{r}_{1,-1} + \beta_2^2 \mathbf{r}'_{2,-1} \mathbf{M} \mathbf{r}_{2,-1} \right] \\ &= \lambda_2^* + O\left(\frac{1}{T^2}\right), \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \lambda_3 &= \frac{1}{T-3} \left[ 2\sigma \beta_1 \frac{2\pi k}{T} \mathbf{v}' \mathbf{M}_y \mathbf{r}_2 - 2\sigma \beta_2 \frac{2\pi k}{T} \mathbf{v}' \mathbf{M}_y \mathbf{r}_1 \right. \\ &\quad + \left( \beta_1 \frac{2\pi k}{T} \right)^2 \mathbf{r}'_2 \mathbf{M}_y \mathbf{r}_2 + \left( \beta_2 \frac{2\pi k}{T} \right)^2 \mathbf{r}'_1 \mathbf{M}_y \mathbf{r}_1 \\ &\quad \left. - \beta_1 \beta_2 \left( \frac{2\pi k}{T} \right)^2 \mathbf{r}'_2 \mathbf{M}_y \mathbf{r}_1 - \beta_1 \beta_2 \left( \frac{2\pi k}{T} \right)^2 \mathbf{r}'_1 \mathbf{M}_y \mathbf{r}_2 \right] \\ &= O\left(\frac{1}{T}\right). \end{aligned} \quad (\text{A.7})$$

Here

$$\begin{aligned} \lambda_1^* &= \frac{1}{T} \left[ \beta_1 \sigma \frac{2\pi k}{T} \mathbf{r}'_2 \mathbf{M} \xi_{y,-1} - \beta_2 \sigma \frac{2\pi k}{T} \mathbf{r}'_1 \mathbf{M} \xi_{y,-1} + \sigma \beta_1 \mathbf{v}' \mathbf{M} \mathbf{r}_{1,-1} \right. \\ &\quad \left. + \beta_1 \beta_2 \frac{2\pi k}{T} \mathbf{r}'_2 \mathbf{M} \mathbf{r}_{2,-1} - \beta_1 \beta_2 \frac{2\pi k}{T} \mathbf{r}'_1 \mathbf{M} \mathbf{r}_{1,-1} + \sigma \beta_2 \mathbf{v}' \mathbf{M} \mathbf{r}_{2,-1} \right], \end{aligned}$$

$$\begin{aligned} \lambda_2^* &= \frac{1}{T^2} \left[ \sigma \beta_1 \xi'_{y,-1} \mathbf{M} \mathbf{r}_{1,-1} + \sigma \beta_2 \xi'_{y,-1} \mathbf{M} \mathbf{r}_{2,-1} + \sigma \beta_1 \mathbf{r}'_{1,-1} \mathbf{M} \xi_{y,-1} \right. \\ &\quad \left. + \beta_1^2 \mathbf{r}'_{1,-1} \mathbf{M} \mathbf{r}_{1,-1} + \beta_2^2 \mathbf{r}'_{2,-1} \mathbf{M} \mathbf{r}_{2,-1} + \sigma \beta_2 \mathbf{r}'_{2,-1} \mathbf{M} \xi_{y,-1} \right]. \end{aligned}$$

From (A.7), it is easily seen that  $\lambda_3$  and  $\frac{1}{2} \left\{ \frac{(2\pi k)^2 \kappa_1^2}{T} + \frac{(2\pi k)^2 \kappa_2^2}{T} \right\}$  are of the same order of magnitude (in probability), because the third and fourth term in  $\lambda_3$  dominates other terms for large  $\kappa$ 's

( $\mathbf{r}'_2 \mathbf{r}_1 = \mathbf{r}'_1 \mathbf{r}_2 = 0$ . See (Hamilton, 1994 p.176.). Substituting (A.5)–(A.7) into (A.4) we obtain

$$\begin{aligned} t^{DF,B} &= -\frac{\frac{\sigma^2 \mathbf{v}' \mathbf{M} \xi_{y,-1}}{T} + \lambda_1}{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M}_y \mathbf{v}}{T-3} + \lambda_3\right)^{1/2} \times \left(\frac{\sigma^2 \xi'_{y,-1} \mathbf{M} \xi_{y,-1}}{T^2} + \lambda_2\right)^{1/2}}, \\ &= \frac{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M} \xi_{y,-1}}{T} + \lambda_1^*\right) + O\left(\frac{1}{T^2}\right)}{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M}_y \mathbf{v}}{T-3} + O\left(\frac{1}{T}\right)\right)^{1/2} \times \left(\left(\frac{\sigma^2 \xi'_{y,-1} \mathbf{M} \xi_{y,-1}}{T^2} + \lambda_2^*\right) + O\left(\frac{1}{T^2}\right)\right)^{1/2}} \\ &= \frac{\frac{\sigma^2 \mathbf{v}' \mathbf{M} \xi_{y,-1}}{T} + \lambda_1^*}{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M}_y \mathbf{v}}{T-3}\right)^{1/2} \times \left(\frac{\sigma^2 \xi'_{y,-1} \mathbf{M} \xi_{y,-1}}{T^2} + \lambda_2^*\right)^{1/2}} \oplus \frac{O\left(\frac{1}{T^2}\right)}{O\left(\frac{1}{\sqrt{T}}\right)}, \end{aligned} \quad (\text{A.8})$$

where the notation  $\oplus$  is adapted from the mediant sums of Farey sequence denoting  $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$ .

The term  $\lambda_1^*$  and  $\lambda_2^*$  can be shown to follow the following asymptotics:

$$\lambda_1^* \Rightarrow 2\pi k \sigma \kappa_1 d_1 - 2\pi k \sigma \kappa_2 d_2 + \sigma \kappa_1 d_3 + \sigma \kappa_2 d_4 + \frac{6\kappa_1 \kappa_2}{\pi k}, \quad (\text{A.9})$$

$$\lambda_2^* \Rightarrow 2\sigma \kappa_1 d_2 + 2\sigma \kappa_2 d_1 + \frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{3\kappa_1^2}{(\pi k)^2}. \quad (\text{A.10})$$

Also, Dickey and Fuller (1979) showed that

$$\begin{aligned} &\frac{\frac{\sigma^2 \mathbf{v}' \mathbf{M} \xi_{y,-1}}{T}}{\left(\frac{\sigma^2 \mathbf{v}' \mathbf{M}_y \mathbf{v}}{T-3}\right)^{1/2} \times \left(\frac{\sigma^2 \xi'_{y,-1} \mathbf{M} \xi_{y,-1}}{T^2}\right)^{1/2}} \\ &\Rightarrow \frac{\int_0^1 W(r)dW(r) - \mathbf{q}'\Psi^{-1}\mathbf{h}}{\left(\int_0^1 W^2(r)dr - \mathbf{h}'\Psi^{-1}\mathbf{h}\right)^{1/2}}. \end{aligned} \quad (\text{A.11})$$

Therefore by substituting (A.9)–(A.11) into (A.8) and applying the continuous mapping theorem, we conclude the proof as in Box III.

The proof of mean case is similar to that of trend case and are omitted to save space. It is available upon request. ■

## Appendix B. Supplementary data

Supplementary material related to this article can be found online at <http://dx.doi.org/10.1016/j.econlet.2017.07.016>.

## References

- Becker, R., Enders, W., Lee, J., 2006. A stationarity test in the presence of an unknown number of smooth breaks. *J. Time Series Anal.* 27, 381–409.
- Dickey, A., Fuller, A., 1979. Distribution of the estimators for autoregressive time series with a unit root. *J. Amer. Statist. Assoc.* 74, 427–431.
- Enders, W., Lee, J., 2012a. A unit root test using a Fourier series to approximate smooth breaks. *Oxford Bull. Econ. Stat.* 74, 574–600.
- Enders, W., Lee, J., 2012b. The flexible Fourier form and Dickey-Fuller type unit root tests. *Econom. Lett.* 117, 196–199.
- Gallant, R., 1981. On the basis in flexible functional forms and an essentially unbiased form: the flexible Fourier form. *J. Econometrics* 15, 211–245.
- Hamilton, J.D., 1994. *Time Series Analysis*. Princeton University Press.

- Lee, C., Wu, J.-L., Yang, L., 2016. A simple panel unit-root test with smooth breaks in the presence of a multifactor error structure. *Oxford Bull. Econ. Stat.* 3, 365–393.
- Leybourne, S., Mills, T., Newbold, P., 1998. Spurious rejections by Dickey-Fuller tests in the presence of a break under the null. *J. Econometrics* 87, 191–203.
- Leybourne, S., Newbold, P., 2000. Behavior of Dickey-Fuller t-tests when there is a break under the null hypothesis. *Econ. J.* 3, 1–15.
- Perron, P., 1989. The great crash, the oil price shock, and the unit root hypothesis. *Econometrica* 57, 1361–1401.
- Perron, P., 2006. Dealing with structural break, In: Palgrave Handbooks of Econometrics: Vol. 1 Econometrics Theory. Mills, T.C. and Patterson, K.. Palgrave Macmillan, Basingstoke, (Chapter 8).