Online Appendix to "A Simple Panel Unit-Root Test with Smooth Breaks in the Presence of a Multifactor Error Structure" by Chingnun Lee, Jyh-Lin Wu and Lixiong Yang (2015)

Abstract

This is a not-for-publication appendix that supplement to the paper "A Simple Panel Unit-Root Test with Smooth Breaks in the Presence of a Multifactor Error Structure" by Chingnun Lee, Jyh-Lin Wu and Lixiong Yang (2015). Appendix A contains mathematical details about the proofs of Theorems (1)-(4) and the sketch of Remarks in the text. Appendix B contains critical values and supplemental tables of our suggested test.

Appendix A: Mathematical Proofs

The following first two Lemmas collect the large sample behavior of the scaled product involving Fourier terms where κ is restricted to be an integer. The third Lemma is about the generalized inverse rule. These Lemmas are used to prove Theorems 1-4 below.

Lemma 1

(L1): $\frac{1}{T}\sum_{t=1}^{T}\sin^2\left(\frac{2\pi\kappa t}{T}\right) \xrightarrow{T} \frac{1}{2}$,

(L2): $\frac{1}{T}\sum_{t=1}^{T}\cos^2\left(\frac{2\pi\kappa t}{T}\right) \xrightarrow{T} \frac{1}{2}$,

(L3): $\sum_{t=1}^{T}\sin(2\pi\kappa t/T)\cos(2\pi\kappa t/T) = 0$,

(L4): $\sum_{t=1}^{T}\sin(2\pi\kappa t/T) = \sum_{t=1}^{T}\cos(2\pi\kappa t/T) = 0$,

(L5): $\Delta\sin(2\pi\kappa t/T) = 2\pi\kappa/T\cos(2\pi\kappa t/T) + o(1)$,

(L6): $\Delta\cos(2\pi\kappa t/T) = -2\pi\kappa/T\sin(2\pi\kappa t/T) + o(1)$.

Lemma 2

Let z_t be a serially correlated and heterogeneously distributed innovation satisfying the following Functional Central Limit Theorem:[19] $W_T(r) = T^{-1/2}\sum_{t=1}^{[Tr]} z_t/\sigma \Rightarrow W(r)$, for $r \in [0,1]$, where $\sigma^2 = \lim_{T\to\infty} T^{-1}E\left(\sum_{t=1}^{T} z_t\right)^2$ and $W(r)$ is a standard Brownian Motion. Then,

(L7): $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} z_t\cos(2\pi\kappa t/T) \xrightarrow{T} \sigma\left[W(1) + 2\pi\kappa\int_0^1\sin(2\pi\kappa r)W(r)dr\right]$,

(L8): $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} z_t\sin(2\pi\kappa t/T) \xrightarrow{T} -2\pi\kappa\sigma\int_0^1\cos(2\pi\kappa r)W(r)dr$,

(L9): $T^{-3/2}\sum_{t=1}^{T}\sum_{s=1}^{t} z_s\sin(2\pi\kappa t/T) \xrightarrow{T} \sigma\left[-2\pi\kappa\int_0^1\cos(2\pi\kappa r)\left[\int_0^r W(s)ds\right]dr\right]$,

(L10): $T^{-3/2}\sum_{t=1}^{T}\sum_{s=1}^{t} z_s\cos(2\pi\kappa t/T) \xrightarrow{T} \sigma\left[\int_0^1 W(s)ds + 2\pi\kappa\int_0^1\sin(2\pi\kappa r)\left[\int_0^r W(s)ds\right]dr\right]$.

[19] See for example, the Theorem 7.18 in White (1999).

Lemma 3

Consider a full column rank $m \times n$ matrix $\boldsymbol{A}$ ($m > n$) and an $n \times n$ non-singular symmetric matrix $\boldsymbol{\Omega}$. Then

(L11): $\boldsymbol{A'}(\boldsymbol{A\Omega A'})^{+}\boldsymbol{A} = \boldsymbol{\Omega}^{-1}$, where $(\boldsymbol{A\Omega A'})^{+}$ is the Moore-Penrose inverse of $\boldsymbol{A\Omega A'}$.[20]

Proof of Lemmas

(L1) and (L2) are given in Becker *et al.* (2006, p.387), (L3) and (L4) are given in Hamilton (1994, p.176), (L5)-(L8) are given in Enders and Lee (2012a, p.594). (L11) is given in Pesaran *et al.* (2013, p.106). To prove (L9), we follow Enders and Lee (2012a) to use the results in Bierens (1994, Lemma 9.6.3): $\sum F(t/T)x_t = F(1)S_T(1) - \int_0^1 f(r)S_T(r)d(r)$, where $S_T(r) = \sum_{t=1}^{[Tr]} x_t$ and $f(r) = F'(r)$. Letting $F(t/T) = \cos(2\pi\kappa t/T)$ and $x_t = \sum_{s=1}^{t} z_s$, and noting that $T^{-3/2}\sum_{t=1}^{[Tr]} x_t \to \sigma \int_0^r W(s)ds$ (Hamilton, p.486), it is straightforward to obtain (L9). Similarly, the results in (L10) can be derived by setting $F(t/T) = \sin(2\pi\kappa t/T)$.

Proof of Theorem 1

The matrix format of (3) under the null hypothesis of $\beta_i = 0$ ($\phi_i = 1$), is:

$$\Delta \boldsymbol{y}_i = \Delta \boldsymbol{D}\boldsymbol{\alpha}_{iy} + \boldsymbol{F}\boldsymbol{\gamma}_{iy} + \boldsymbol{\varepsilon}_{iy},\ i = 1, ..., N. \tag{40}$$

Under the null hypothesis of $\beta_i = 0$, we have $\boldsymbol{B}_i = \boldsymbol{0}$, $\boldsymbol{C}_i = \boldsymbol{0}$ and $\ddot{\boldsymbol{A}}_i = \boldsymbol{A}_i$ in (10). By substituting $\boldsymbol{B}_i = \boldsymbol{0}$, $\boldsymbol{C}_i = \boldsymbol{0}$, and $\ddot{\boldsymbol{A}}_i = \boldsymbol{A}_i$ into (12), we obtain:

$$\boldsymbol{F} = (\Delta \bar{\boldsymbol{z}} - \Delta \boldsymbol{D}\bar{\boldsymbol{A}}' - \bar{\boldsymbol{\varepsilon}})\bar{\boldsymbol{\Gamma}}(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}})^{-1}. \tag{41}$$

By substituting (41) into (40), the matrix format of y_{it} in difference form is:

$$\begin{aligned} \triangle \boldsymbol{y}_i &= \Delta \boldsymbol{D}\boldsymbol{\alpha}_{iy} + (\Delta \overline{\boldsymbol{z}} - \Delta \boldsymbol{D}\overline{\boldsymbol{A}}')\boldsymbol{\delta}_i + \boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i, \\ &= \Delta \boldsymbol{D}\boldsymbol{\alpha}_i + \Delta \overline{\boldsymbol{z}}'\boldsymbol{\delta}_i + \boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i, \end{aligned} \tag{42}$$

where $\boldsymbol{\alpha}_i = \boldsymbol{\alpha}_{iy} - \overline{\boldsymbol{A}}'\boldsymbol{\delta}_i$ and $\boldsymbol{\delta}_i = \bar{\boldsymbol{\Gamma}}(\bar{\boldsymbol{\Gamma}}'\bar{\boldsymbol{\Gamma}})^{-1}\boldsymbol{\gamma}_{iy}$. Denoting that

$$\boldsymbol{v}_i = (\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)/\sigma_i, \tag{43}$$

so[21]

$$\boldsymbol{M}_z \Delta \boldsymbol{y}_i = \sigma_i \boldsymbol{M}_z \boldsymbol{v}_i, \tag{44}$$

$$\boldsymbol{M}_{i,z} \Delta \boldsymbol{y}_i = \sigma_i \boldsymbol{M}_{i,z} \boldsymbol{v}_i. \tag{45}$$

[20] For a full column rank $m \times n$ matrix $\boldsymbol{U}$ ($m > n$), if there exists an $n \times m$ (here m can be equal to n) matrix, $\boldsymbol{X}$, satisfying the following conditions: (a) $\boldsymbol{UXU} = \boldsymbol{U}$, (b) $\boldsymbol{XUX} = \boldsymbol{X}$, (c) $(\boldsymbol{UX})' = \boldsymbol{UX}$, and (d) $(\boldsymbol{XU})' = \boldsymbol{XU}$, then $\boldsymbol{X}$ is called the Moore-Penrose inverse of $\boldsymbol{U}$, denoted as $\boldsymbol{U}^{+}$. It is well-known that $\boldsymbol{U}^{+}$ exists and is unique for any $m \times n$ matrix $\boldsymbol{U}$.

[21] Because $\boldsymbol{\Upsilon}_i = \Delta \boldsymbol{\Upsilon}_i + \boldsymbol{\Upsilon}_{i,-1}$, for $i = 1, 2$. $\boldsymbol{M}_z \boldsymbol{\Upsilon}_i = \boldsymbol{M}_z \Delta \boldsymbol{\Upsilon}_i + \boldsymbol{M}_z \boldsymbol{\Upsilon}_{i,-1} = \boldsymbol{0}$. Hence $\boldsymbol{M}_z \Delta \boldsymbol{\Upsilon}_i = \boldsymbol{M}_z \boldsymbol{\Upsilon}_{i,-1} = \boldsymbol{0}$.

By recursively substituting (42), we obtain y_{it} in level form as:

$$\begin{aligned} y_{i,t-1} - y_{i0} &= (\boldsymbol{d}_{t-1} - \boldsymbol{d}_0)(\boldsymbol{\alpha}_{iy} - \bar{\boldsymbol{A}}'\boldsymbol{\delta}_i) + (\bar{\boldsymbol{z}}_{t-1} - \bar{\boldsymbol{z}}_0)'\boldsymbol{\delta}_i + s_{iy,t-1} - \\ & \quad N^{-1}\left(\sum_{i=1}^{N} s_{iy,t-1},\ \sum_{i=1}^{N} s_{ix,t-1}^{1},\ \sum_{i=1}^{N} s_{ix,t-1}^{2}, ..., \sum_{i=1}^{N} s_{ix,t-1}^{k}\right)' \boldsymbol{\delta}_i. \end{aligned} \tag{46}$$

By assuming that $\boldsymbol{d}_0 = \boldsymbol{0}$, the matrix format of (46) is given as follows:

$$\boldsymbol{y}_{i,-1} = \overset{\circ}{y}_{i0}\,\boldsymbol{\tau} + \boldsymbol{D}_{-1}(\boldsymbol{\alpha}_{iy} - \bar{\boldsymbol{A}}'\boldsymbol{\delta}_i) + \bar{\boldsymbol{z}}_{-1}\boldsymbol{\delta}_i + \boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i, \tag{47}$$

where $y_{i0}^{\circ} = y_{i0} - \bar{\boldsymbol{z}}_0'\boldsymbol{\delta}_i$, $\boldsymbol{D}_{-1} = (\boldsymbol{0}, \boldsymbol{d}_1, ..., \boldsymbol{d}_{T-1})'$, $\boldsymbol{s}_{iy,-1} = (0, s_{iy,1}, s_{iy,2}, ..., s_{iy,T-1})'$, and $\bar{\boldsymbol{S}}_{-1} = N^{-1}\sum_{i=1}^{N}\boldsymbol{S}_{i,-1}$ with $\boldsymbol{S}_{i,-1} = (0, \boldsymbol{s}_{i,1}, \boldsymbol{s}_{i,2}, ..., \boldsymbol{s}_{i,T-1})'$. Let $\boldsymbol{\xi}_{i,-1} = (\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)/\sigma_i$, then

$$\boldsymbol{M}_z\boldsymbol{y}_{i,-1} = \sigma_i\boldsymbol{M}_z\boldsymbol{\xi}_{i,-1}. \tag{48}$$

By plugging (44),(45) and (48) into (15), the t-statistic can be expressed as:

$$t_i(N,T) = \frac{\Delta\boldsymbol{y}'_i\boldsymbol{M}_z\boldsymbol{y}_{i,-1}}{\hat{\sigma}_i\left(\boldsymbol{y}_{i,-1}'\boldsymbol{M}_z\boldsymbol{y}_{i,-1}\right)^{1/2}} = \frac{\frac{\boldsymbol{v}_i'\boldsymbol{M}_z\boldsymbol{\xi}_{i,-1}}{T}}{\left(\frac{\boldsymbol{v}_i'\boldsymbol{M}_{i,z}\boldsymbol{v}_i}{T-2k-6}\right)^{1/2}\left(\frac{\boldsymbol{\xi}_{i,-1}'\boldsymbol{M}_z\boldsymbol{\xi}_{i,-1}}{T^2}\right)^{1/2}}. \tag{49}$$

The numerator of the $t_i(N,T)$ in (49) can be rewritten as:

$$\frac{\boldsymbol{v}_i'\boldsymbol{M}_z\xi_{i,-1}}{T} = \frac{\boldsymbol{v}_i'\boldsymbol{\xi}_{i,-1}}{T} - (\boldsymbol{v}_i'\boldsymbol{Z}\boldsymbol{B})(\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{B})^{-1}\frac{\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{\xi}_{i,-1}}{T}, \tag{50}$$

where $\boldsymbol{B}_{(2k+5)\times(2k+5)} = diag\left[T^{-1/2}\boldsymbol{I}_{k+4},\ T^{-1}\boldsymbol{I}_{k+1}\right]$. Furthermore, it can be shown that

$$(\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{v}_i)_{(2k+5)\times 1} = \left[\frac{\boldsymbol{v}_i'\Delta\bar{\boldsymbol{z}}}{\sqrt{T}}, \frac{\boldsymbol{v}_i'\boldsymbol{\tau}}{\sqrt{T}}, \frac{\boldsymbol{v}_i'\boldsymbol{\Upsilon}_1}{\sqrt{T}}, \frac{\boldsymbol{v}_i'\boldsymbol{\Upsilon}_2}{\sqrt{T}}, \frac{\boldsymbol{v}_i'\bar{\boldsymbol{z}}_{-1}}{T}\right]', \tag{51}$$

$$(\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{\xi}_{i,-1})_{(2k+5)\times 1} = \left[\frac{\boldsymbol{\xi}_{i,-1}'\Delta\bar{\boldsymbol{z}}}{\sqrt{T}}, \frac{\boldsymbol{\xi}_{i,-1}'\boldsymbol{\tau}}{\sqrt{T}}, \frac{\boldsymbol{\xi}_{i,-1}'\boldsymbol{\Upsilon}_1}{\sqrt{T}}, \frac{\boldsymbol{\xi}_{i,-1}'\boldsymbol{\Upsilon}_2}{\sqrt{T}}, \frac{\boldsymbol{\xi}_{i,-1}'\bar{\boldsymbol{z}}_{-1}}{T}\right]', \tag{52}$$

$$\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{B} = \begin{bmatrix} \frac{\Delta\bar{\boldsymbol{z}}'\Delta\bar{\boldsymbol{z}}}{T} & \frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\tau}}{T} & \frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\Upsilon}_1}{T} & \frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\Upsilon}_2}{T} & \frac{\Delta\bar{\boldsymbol{z}}'\bar{\boldsymbol{z}}_{-1}}{T^{3/2}} \\ \frac{\boldsymbol{\tau}'\Delta\bar{\boldsymbol{z}}}{T} & \frac{\boldsymbol{\tau}'\boldsymbol{\tau}}{T} & \frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_1}{T} & \frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2}{T} & \frac{\boldsymbol{\tau}'\bar{\boldsymbol{z}}_{-1}}{T^{3/2}} \\ \frac{\boldsymbol{\Upsilon}_1'\Delta\bar{\boldsymbol{z}}}{T} & \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\tau}}{T} & \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\Upsilon}_1}{T} & \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\Upsilon}_2}{T} & \frac{\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{z}}_{-1}}{T^{3/2}} \\ \frac{\boldsymbol{\Upsilon}_2'\Delta\bar{\boldsymbol{z}}}{T} & \frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\tau}}{T} & \frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\Upsilon}_1}{T} & \frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\Upsilon}_2}{T} & \frac{\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{z}}_{-1}}{T^{3/2}} \\ \frac{\bar{\boldsymbol{z}}'_{-1}\Delta\bar{\boldsymbol{z}}}{T^{3/2}} & \frac{\bar{\boldsymbol{z}}'_{-1}\boldsymbol{\tau}}{T^{3/2}} & \frac{\bar{\boldsymbol{z}}'_{-1}\boldsymbol{\Upsilon}_1}{T^{3/2}} & \frac{\bar{\boldsymbol{z}}'_{-1}\boldsymbol{\Upsilon}_2}{T^{3/2}} & \frac{\bar{\boldsymbol{z}}'_{-1}\bar{\boldsymbol{z}}_{-1}}{T^2} \end{bmatrix}. \tag{53}$$

Under then null hypothesis of $\beta_i = 0$, the matrix format of (11) is (in the case of $\boldsymbol{d}_t = (1, \sin(2\pi\kappa t/T), \cos(2\pi\kappa t/T))'$):

$$\Delta\bar{\boldsymbol{z}} = \Delta\boldsymbol{D}\bar{\boldsymbol{A}}' + \boldsymbol{F}\bar{\boldsymbol{\Gamma}}' + \bar{\varepsilon} = \Delta\boldsymbol{\Upsilon}_1\bar{\boldsymbol{\alpha}}_{i,1}' + \Delta\boldsymbol{\Upsilon}_2\bar{\boldsymbol{\alpha}}_{i,2}' + \boldsymbol{F}\bar{\boldsymbol{\Gamma}}' + \bar{\varepsilon}. \tag{54}$$

By using the same recursive method in deriving (47), the lagged level cross-sectional average, $\bar{\boldsymbol{z}}_{-1}$, can be expressed as:

$$\bar{\boldsymbol{z}}_{-1} = \bar{\boldsymbol{z}}_0 + \boldsymbol{D}_{-1}\bar{\boldsymbol{A}}' + \boldsymbol{s}_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{S}}_{-1} = \bar{\boldsymbol{z}}_0 + \boldsymbol{\Upsilon}_{1,-1}\bar{\boldsymbol{\alpha}}'_{i,1} + \boldsymbol{\Upsilon}_{2,-1}\bar{\boldsymbol{\alpha}}'_{i,2} + \boldsymbol{s}_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{S}}_{-1}, \tag{55}$$

where $\boldsymbol{A}_i = [\boldsymbol{\alpha}_{i,1},\ \boldsymbol{\alpha}_{i,2}]$. By using (54), (55), $\boldsymbol{v}_i = (\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)/\sigma_i$, and $\boldsymbol{\xi}_{i,-1} = (\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)/\sigma_i$, we express the elements involving cross-sectional averages in the numerator of $t_i(N,T)$ ((50)-(53)) as follows:

$$\frac{\boldsymbol{v}_i'\boldsymbol{\xi}_{i,-1}}{T} = \frac{(\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1} - \boldsymbol{\varepsilon}_{iy}'\bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i - \boldsymbol{\delta}_i'\bar{\boldsymbol{\varepsilon}}'\boldsymbol{s}_{iy,-1} + \boldsymbol{\delta}_i'\bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i^2 T} \tag{56}$$

$$\frac{\bar{\boldsymbol{z}}'_{-1}\boldsymbol{v}_i}{T} = \frac{\bar{\boldsymbol{z}}'_0(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i T} + \frac{\bar{\boldsymbol{A}}\boldsymbol{D}'_{-1}(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i T} + \frac{\bar{\boldsymbol{\Gamma}}\boldsymbol{s}'_{f,-1}(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i T} + \frac{\bar{\boldsymbol{S}}'_{-1}(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i T}, \tag{57}$$

$$\frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{v}_i}{\sigma_i\sqrt{T}} = \frac{\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}} + \frac{\bar{\boldsymbol{\Gamma}}\boldsymbol{F}'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}} + \frac{\bar{\boldsymbol{\varepsilon}}'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{58}$$

$$\frac{\boldsymbol{\tau}'\boldsymbol{v}_i}{\sqrt{T}} = \frac{\boldsymbol{\tau}'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{59}$$

$$\frac{\boldsymbol{\Upsilon}_1'\boldsymbol{v}_i}{\sqrt{T}} = \frac{\boldsymbol{\Upsilon}_1'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{60}$$

$$\frac{\boldsymbol{\Upsilon}_2'\boldsymbol{v}_i}{\sqrt{T}} = \frac{\boldsymbol{\Upsilon}_2'(\boldsymbol{\varepsilon}_{iy} - \bar{\boldsymbol{\varepsilon}}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{61}$$

$$\frac{\bar{\boldsymbol{z}}'_{-1}\boldsymbol{\xi}_{i,-1}}{T} = \frac{\bar{\boldsymbol{z}}'_0(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i T} + \frac{\bar{\boldsymbol{A}}\boldsymbol{D}'_{-1}(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i T} + \frac{\bar{\Gamma}\boldsymbol{s}'_{f,-1}(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i T}, \tag{62}$$

$$\frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\xi}_{i,-1}}{\sqrt{T}} = \frac{\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}} + \frac{\bar{\boldsymbol{\Gamma}}\boldsymbol{F}'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}} + \frac{\bar{\boldsymbol{\varepsilon}}'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{63}$$

$$\frac{\boldsymbol{\tau}'\boldsymbol{\xi}_{i,-1}}{\sqrt{T}} = \frac{\boldsymbol{\tau}'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{64}$$

$$\frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\xi}_{i,-1}}{\sqrt{T}} = \frac{\boldsymbol{\Upsilon}_1'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{65}$$

$$\frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\xi}_{i,-1}}{\sqrt{T}} = \frac{\boldsymbol{\Upsilon}_2'(\boldsymbol{s}_{iy,-1} - \bar{\boldsymbol{S}}_{-1}\boldsymbol{\delta}_i)}{\sigma_i\sqrt{T}}, \tag{66}$$

$$\frac{\Delta\bar{\boldsymbol{z}}'\Delta\bar{\boldsymbol{z}}}{T} = \frac{1}{T}(\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\Delta\boldsymbol{D}\bar{\boldsymbol{A}}' + \bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\boldsymbol{F}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\Gamma}}\boldsymbol{F}'\Delta\boldsymbol{D}\bar{\boldsymbol{A}}' + \bar{\boldsymbol{\Gamma}}\boldsymbol{F}'\boldsymbol{F}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{\Gamma}}\boldsymbol{F}'\bar{\boldsymbol{\varepsilon}} + \bar{\boldsymbol{\varepsilon}}'\Delta\boldsymbol{D}\bar{\boldsymbol{A}}' + \bar{\boldsymbol{\varepsilon}}'\boldsymbol{F}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{\varepsilon}}'\bar{\boldsymbol{\varepsilon}}), \tag{67}$$

$$\frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\tau}}{T} = \frac{1}{T}(\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\boldsymbol{\tau} + \bar{\Gamma}\boldsymbol{F}'\boldsymbol{\tau} + \bar{\boldsymbol{\varepsilon}}'\boldsymbol{\tau}), \tag{68}$$

$$\frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\Upsilon}_1}{T} = \frac{1}{T}(\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\boldsymbol{\Upsilon}_1 + \bar{\boldsymbol{\Gamma}}\boldsymbol{F}'\boldsymbol{\Upsilon}_1 + \bar{\boldsymbol{\varepsilon}}'\boldsymbol{\Upsilon}_1), \frac{\Delta\bar{\boldsymbol{z}}'\boldsymbol{\Upsilon}_2}{T} \tag{69}$$

$$= \frac{1}{T}(\bar{\boldsymbol{A}}\Delta\boldsymbol{D}'\boldsymbol{\Upsilon}_2 + \bar{\boldsymbol{\Gamma}}\boldsymbol{F}'\boldsymbol{\Upsilon}_2 + \bar{\boldsymbol{\varepsilon}}'\boldsymbol{\Upsilon}_2), \tag{70}$$

$$\frac{\Delta \bar{z}' \bar{z}_{-1}}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{A}\Delta D' \bar{z}_0 + \bar{A}\Delta D' D_{-1}\bar{\mathbf{A}}' + \bar{A}\Delta D' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{A}\Delta D' \bar{S}_{-1}$$
$$+\bar{\boldsymbol{\Gamma}} F' \bar{z}_0 + \bar{\boldsymbol{\Gamma}} F' D_{-1}\bar{\mathbf{A}}' + \bar{\boldsymbol{\Gamma}} F' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{\Gamma}} F' \bar{S}_{-1} + \bar{\varepsilon}' \bar{z}_0 + \bar{\varepsilon}' D_{-1}\bar{\mathbf{A}}'$$
$$+\bar{\varepsilon}' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{\varepsilon}' \bar{S}_{-1}), \quad (71)$$

$$\frac{\tau' \Delta \bar{z}}{T} = \frac{1}{T}(\tau' \Delta D \bar{A}' + \tau' F \bar{\boldsymbol{\Gamma}}' + \tau' \bar{\varepsilon}), \quad (72)$$

$$\frac{\tau' \bar{z}_{-1}}{T^{3/2}} = \frac{1}{T^{3/2}}(\tau' \bar{z}_0 + \tau' D_{-1}\bar{\mathbf{A}}' + \tau' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \tau' \bar{S}_{-1}), \quad (73)$$

$$\frac{\boldsymbol{\Upsilon}_1' \Delta \bar{z}}{T} = \frac{1}{T}(\boldsymbol{\Upsilon}_1' \Delta D \bar{A}' + \boldsymbol{\Upsilon}_1' F \bar{\boldsymbol{\Gamma}}' + \boldsymbol{\Upsilon}_1' \bar{\varepsilon}), \quad (74)$$

$$\frac{\boldsymbol{\Upsilon}_1' \bar{z}_{-1}}{T^{3/2}} = \frac{1}{T^{3/2}}(\boldsymbol{\Upsilon}_1' \bar{z}_0 + \boldsymbol{\Upsilon}_1' D_{-1}\bar{\mathbf{A}}' + \boldsymbol{\Upsilon}_1' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \boldsymbol{\Upsilon}_1' \bar{S}_{-1}), \quad (75)$$

$$\frac{\boldsymbol{\Upsilon}_2' \Delta \bar{z}}{T} = \frac{1}{T}(\boldsymbol{\Upsilon}_2' \Delta D \bar{A}' + \boldsymbol{\Upsilon}_2' F \bar{\boldsymbol{\Gamma}}' + \boldsymbol{\Upsilon}_2' \bar{\varepsilon}), \quad (76)$$

$$\frac{\boldsymbol{\Upsilon}_2' \bar{z}_{-1}}{T^{3/2}} = \frac{1}{T^{3/2}}(\boldsymbol{\Upsilon}_2' \bar{z}_0 + \boldsymbol{\Upsilon}_2' D_{-1}\bar{\mathbf{A}}' + \boldsymbol{\Upsilon}_2' s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \boldsymbol{\Upsilon}_2' \bar{S}_{-1}), \quad (77)$$

$$\frac{\bar{z}'_{-1} \Delta \bar{z}}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{z}'_0 \Delta D \bar{A}' + \bar{z}'_0 F \bar{\boldsymbol{\Gamma}}' + \bar{z}'_0 \bar{\varepsilon} + \bar{A} D'_{-1} \Delta D \bar{A}' + \bar{A} D'_{-1} F \bar{\boldsymbol{\Gamma}}' + \bar{A} D'_{-1} \bar{\varepsilon}$$
$$+\bar{\boldsymbol{\Gamma}} s'_{f,-1} \Delta D \bar{A}' + \bar{\boldsymbol{\Gamma}} s'_{f,-1} F \bar{\boldsymbol{\Gamma}}' + \bar{\boldsymbol{\Gamma}} s'_{f,-1} \bar{\varepsilon} + \bar{S}'_{-1} \Delta D \bar{A}'$$
$$+\bar{S}'_{-1} F \bar{\boldsymbol{\Gamma}}' + \bar{S}'_{-1} \bar{\varepsilon}), \quad (78)$$

$$\frac{\bar{z}'_{-1} \tau}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{z}'_0 \tau + \bar{A} D'_{-1} \tau + \bar{\boldsymbol{\Gamma}} s'_{f,-1} \tau + \bar{S}'_{-1} \tau), \quad (79)$$

$$\frac{\bar{z}'_{-1} \boldsymbol{\Upsilon}_1}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{z}'_0 \boldsymbol{\Upsilon}_1 + \bar{A} D'_{-1} \boldsymbol{\Upsilon}_1 + \bar{\boldsymbol{\Gamma}} s'_{f,-1} \boldsymbol{\Upsilon}_1 + \bar{S}'_{-1} \boldsymbol{\Upsilon}_1), \quad (80)$$

$$\frac{\bar{z}'_{-1} \boldsymbol{\Upsilon}_2}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{z}'_0 \boldsymbol{\Upsilon}_2 + \bar{A} D'_{-1} \boldsymbol{\Upsilon}_2 + \bar{\boldsymbol{\Gamma}} s'_{f,-1} \boldsymbol{\Upsilon}_2 + \bar{S}'_{-1} \boldsymbol{\Upsilon}_2), \quad (81)$$

$$\frac{\bar{z}'_{-1} \bar{z}_{-1}}{T^{3/2}} = \frac{1}{T^{3/2}}(\bar{z}'_0 \bar{z}_0 + \bar{z}'_0 D_{-1}\bar{\mathbf{A}}' + \bar{z}'_0 s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{z}'_0 \bar{S}_{-1} + \bar{A} D'_{-1} \bar{z}_0 + \bar{A} D'_{-1} D_{-1}\bar{\mathbf{A}}'$$
$$+\bar{A} D'_{-1} s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{A} D'_{-1} \bar{S}_{-1} + \bar{\boldsymbol{\Gamma}} s'_{f,-1} \bar{z}_0 + \bar{\boldsymbol{\Gamma}} s'_{f,-1} D_{-1}\bar{\mathbf{A}}' + \bar{\boldsymbol{\Gamma}} s'_{f,-1} s_{f,-1}\bar{\boldsymbol{\Gamma}}'$$
$$+\bar{\boldsymbol{\Gamma}} s'_{f,-1} \bar{S}_{-1} + \bar{S}'_{-1} \bar{z}_0 + \bar{S}'_{-1} D_{-1}\bar{\mathbf{A}}' + \bar{S}'_{-1} s_{f,-1}\bar{\boldsymbol{\Gamma}}' + \bar{S}'_{-1} \bar{S}_{-1}). \quad (82)$$

- $N \to \infty$ and T fixed

As in Pesaran *et al.* (2009), all stochastic terms involving averaging over i go to zero as $N \to \infty$. Let z_{it} be expressed as the deviation from its cross-sectional mean of the initial observations then $\bar{z}_0 = \mathbf{0}$. By substituting (57)-(61) into (51), we have:

$$BZ' v_i \xrightarrow{N} \left[\begin{array}{ccccc} (A^* \Delta D' + \boldsymbol{\Gamma}^* F') \frac{\varepsilon_{iy}}{\sigma_i \sqrt{T}} & \frac{\tau' \varepsilon_{iy}}{\sigma_i \sqrt{T}} & \frac{\boldsymbol{\Upsilon}_1' \varepsilon_{iy}}{\sigma_i \sqrt{T}} & \frac{\boldsymbol{\Upsilon}_2' \varepsilon_{iy}}{\sigma_i \sqrt{T}} & (A^* D'_{-1} + \boldsymbol{\Gamma}^* s'_{f,-1}) \frac{\varepsilon_{iy}}{\sigma_i T} \end{array} \right]', \quad (83)$$

where

$$A^* = \lim_{N \to \infty} \overline{A} \equiv \left[\begin{array}{cc} \boldsymbol{\alpha}_1^* & \boldsymbol{\alpha}_2^* \end{array} \right] \text{ and } \boldsymbol{\Gamma}^* = \lim_{N \to \infty} \overline{\boldsymbol{\Gamma}}. \quad (84)$$

Based on Lemmas (L5) and (L6), we have $\boldsymbol{\Upsilon}_1 - \boldsymbol{\Upsilon}_{1,-1} = \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2$ and $\boldsymbol{\Upsilon}_2 - \boldsymbol{\Upsilon}_{2,-1} = -\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1$. The terms $\boldsymbol{A}^*\Delta\boldsymbol{D}'$ and $\boldsymbol{A}^*\boldsymbol{D}'_{-1}$ in the right-hand side of (83) are:

$$\boldsymbol{A}^*\Delta\boldsymbol{D}' = \begin{bmatrix} \boldsymbol{\alpha}_1^* & \boldsymbol{\alpha}_2^* \end{bmatrix} \begin{bmatrix} \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2' \\ -\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1' \end{bmatrix} = \frac{2\pi\kappa}{T}\boldsymbol{\alpha}_1^*\boldsymbol{\Upsilon}_2' - \frac{2\pi\kappa}{T}\boldsymbol{\alpha}_2^*\boldsymbol{\Upsilon}_1', \tag{85}$$

$$\begin{aligned} \boldsymbol{A}^*\boldsymbol{D}'_{-1} &= \begin{bmatrix} \boldsymbol{\alpha}_1^* & \boldsymbol{\alpha}_2^* \end{bmatrix} \begin{bmatrix} \boldsymbol{\Upsilon}_{1,-1}' \\ \boldsymbol{\Upsilon}_{2,-1}' \end{bmatrix} = \boldsymbol{\alpha}_1^*(\boldsymbol{\Upsilon}_1' - \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2') + \boldsymbol{\alpha}_2^*(\boldsymbol{\Upsilon}_2' + \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1') \\ &= \boldsymbol{\alpha}_1^{**}\boldsymbol{\Upsilon}_1' + \boldsymbol{\alpha}_2^{**}\boldsymbol{\Upsilon}_2', \end{aligned} \tag{86}$$

where $\boldsymbol{\alpha}_1^{**} = \boldsymbol{\alpha}_1^* + \boldsymbol{\alpha}_2^*\frac{2\pi\kappa}{T}$ and $\boldsymbol{\alpha}_2^{**} = \boldsymbol{\alpha}_2^* - \boldsymbol{\alpha}_1^*\frac{2\pi\kappa}{T}$. By substituting (85) and (86) into (83), we obtain:

$$\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{v}_i \xrightarrow{N} \boldsymbol{\Pi}_T^*\boldsymbol{q}_{iT}, \tag{87}$$

where

$$\boldsymbol{\Pi}_T^* = \begin{bmatrix} \boldsymbol{\Gamma}^* & 0 & -\frac{2\pi\kappa}{T}\boldsymbol{\alpha}_2^* & \frac{2\pi\kappa}{T}\boldsymbol{\alpha}_1^* & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{T}}\boldsymbol{\alpha}_1^{**} & \frac{1}{\sqrt{T}}\boldsymbol{\alpha}_2^{**} & \boldsymbol{\Gamma}^* \end{bmatrix}, \text{ and } \boldsymbol{q}_{iT} = \begin{bmatrix} \frac{\boldsymbol{F}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \\ \frac{\boldsymbol{\tau}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \\ \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \\ \frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \\ \frac{\boldsymbol{s}_{f,-1}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i T} \end{bmatrix}.$$

Similarly, by substituting (62)-(66) into (52) and (67)-(82) into (53), we obtain:

$$\frac{\boldsymbol{B}\boldsymbol{Z}'\boldsymbol{\xi}_{i,-1}}{T} \xrightarrow{N} \boldsymbol{\Pi}_T^*\boldsymbol{h}_{iT} \text{ and } \boldsymbol{B}\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{B} \xrightarrow{N} \boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*, \tag{88}$$

where $\boldsymbol{h}_{iT}$ and $\boldsymbol{\Psi}_{fT}$ are defined in (19) and (20), respectively. By combining (56), (87) and (88), the numerator of the $t_i(N,T)$ in (50) is shown to have the following limiting distribution as $N \to \infty$:

$$\frac{\boldsymbol{v}_i'\boldsymbol{M}_z\boldsymbol{\xi}_{i,-1}}{T} \xrightarrow{N} \frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - (\boldsymbol{q}_{iT}'\boldsymbol{\Pi}_T'^*)(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)^+(\boldsymbol{\Pi}_T^*\boldsymbol{h}_{iT}). \tag{89}$$

Although $\boldsymbol{\Pi}_T^*$ is a full rank matrix and $\boldsymbol{\Psi}_{fT}$ is a nonsingular matrix, $(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)$ may be a singular matrix.[22] Therefore, the general inverse is used in the right-hand side of (89). Same argument holds for $(\boldsymbol{\Theta}_T^*\boldsymbol{\Xi}_{iT}\boldsymbol{\Theta}_T'^*)$ below.

Similarly, the elements in the denominator of the t-statistic as $N \to \infty$ are distributed as:

$$\frac{\boldsymbol{v}_i'\boldsymbol{M}_{i,z}\boldsymbol{v}_i}{T-2k-6} \xrightarrow{N} \frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i^2(T-2k-6)} - \frac{(\boldsymbol{d}_{iT}'\boldsymbol{\Theta}_T'^*)(\boldsymbol{\Theta}_T^*\boldsymbol{\Xi}_{iT}\boldsymbol{\Theta}_T'^*)^+(\boldsymbol{\Theta}_T^*\boldsymbol{d}_{iT})}{T-2k-6}, \tag{90}$$

$$\frac{\boldsymbol{\xi}_{i,-1}'\boldsymbol{M}_z\boldsymbol{\xi}_{i,-1}}{T^2} \xrightarrow{N} \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} - (\boldsymbol{h}_{iT}'\boldsymbol{\Pi}_T'^*)(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)^+(\boldsymbol{\Pi}_T^*\boldsymbol{h}_{iT}), \tag{91}$$

[22] For example, let $\boldsymbol{\Pi}_T^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\boldsymbol{\Psi}_{fT} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, then $\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^* = \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a singular matrix.

where

$$\boldsymbol{\Xi}_{iT} = \begin{bmatrix} & & \boldsymbol{\Psi}_{fT} & & & \begin{matrix} \frac{\boldsymbol{F}'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} \\ \frac{\boldsymbol{\tau}'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} \\ \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} \\ \frac{\boldsymbol{\Upsilon}_2'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} \\ \frac{\boldsymbol{s}_{f,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i T^2} \end{matrix} \\ \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{F}}{\sigma_i T^{3/2}} & \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{\tau}}{\sigma_i T^{3/2}} & \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{\Upsilon}_1}{\sigma_i T^{3/2}} & \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{\Upsilon}_2}{\sigma_i T^{3/2}} & \frac{\boldsymbol{s}_{i,-1}'\boldsymbol{s}_{f,-1}}{\sigma_i T^2} & \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} \end{bmatrix}, \tag{92}$$

and

$$\boldsymbol{\Theta}_T^* = \begin{bmatrix} & & \boldsymbol{\Pi}_T^* & & & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ 0 & \frac{1}{\sqrt{T}} y_{i0} & \frac{1}{\sqrt{T}}\boldsymbol{\alpha}_{i,1} & \frac{1}{\sqrt{T}}\boldsymbol{\alpha}_{i,2} & \boldsymbol{\gamma}_{iy}' & 1 \end{bmatrix}, \ \boldsymbol{d}_{iT} \equiv \begin{bmatrix} \boldsymbol{q}_{iT}' & \frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i T} \end{bmatrix}'. \tag{93}$$

By substituting (89)-(91) into (49), we obtain:

$$t_i(N,T) \xrightarrow{N} \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - (\boldsymbol{q}_{iT}'\boldsymbol{\Pi}_T'^*)(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)^+(\boldsymbol{\Pi}_T^*\boldsymbol{h}_{iT})}{J_1 \times J_2}, \tag{94}$$

where

$$J_1 = \left(\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i^2(T-2k-6)} - \frac{(\boldsymbol{d}_{iT}'\boldsymbol{\Theta}_T'^*)(\boldsymbol{\Theta}_T^*\boldsymbol{\Xi}_{iT}\boldsymbol{\Theta}_T'^*)^+(\boldsymbol{\Theta}_T^*\boldsymbol{d}_{iT})}{T-2k-6}\right)^{1/2},$$

$$J_2 = \left(\frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} - (\boldsymbol{h}_{iT}'\boldsymbol{\Pi}_T'^*)(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)^+(\boldsymbol{\Pi}_T^*\boldsymbol{h}_{iT})\right)^{1/2}.$$

$\boldsymbol{\Pi}_T^*$ and $\boldsymbol{\Theta}_T^*$ are full column rank matrices based on the rank condition of Assumption 5, and $\boldsymbol{\Psi}_{fT}$ and $\boldsymbol{\Xi}_{iT}$ are nonsingular because of Assumptions 1, 2 and 3. Based on Lemma 3, we obtain $\boldsymbol{\Pi}_T'^*(\boldsymbol{\Pi}_T^*\boldsymbol{\Psi}_{fT}\boldsymbol{\Pi}_T'^*)^+\boldsymbol{\Pi}_T^* = \boldsymbol{\Psi}_{fT}^{-1}$, and $\boldsymbol{\Theta}_T'^*(\boldsymbol{\Theta}_T^*\boldsymbol{\Xi}_{iT}\boldsymbol{\Theta}_T'^*)^+\boldsymbol{\Theta}_T^* = \boldsymbol{\Xi}_{iT}^{-1}$. Therefore, the limiting distribution of the $t_i(N,T)$ statistic can be simplified as:

$$t_i(N,T) \xrightarrow{N} \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \boldsymbol{q}_{iT}'\boldsymbol{\Psi}_{fT}^{-1}\boldsymbol{h}_{iT}}{\left(\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i^2(T-2k-6)} - \frac{\boldsymbol{d}_{iT}'\boldsymbol{\Xi}_{iT}^{-1}\boldsymbol{d}_{iT}}{T-2k-6}\right)^{1/2} \times \left(\frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} - \boldsymbol{h}_{iT}'\boldsymbol{\Psi}_{fT}^{-1}\boldsymbol{h}_{iT}\right)^{1/2}}, \tag{95}$$

which does not depend on nuisance parameters since $\varepsilon_{iyt}/\sigma_i$ is independently distributed as $i.i.d.(0,1)$ by Assumption 1. This completes the proof of Theorem 1. ■

Proof of Theorem 2

● Sequential Asymptotic: $N \to \infty$ then $T \to \infty$

Given the above results obtained from T fixed and $N \to \infty$, we now allow T to approach infinity for completing the proof of sequential limit. By using Lemmas 1 and 2 as well as the results in Hamilton (1994, P.486) and Pesaran *et al.* (2013), the terms in the numerator of the limiting distribution of the t-statistic in (95) are distributed, when $T \to \infty$, as:

$$
\begin{aligned}
T^{-1/2}\sum_{t=1}^{T}\varepsilon_{iyt}/\sigma_i &\xrightarrow{T} W_i(1), \quad \frac{\boldsymbol{F}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \xrightarrow{T} \boldsymbol{\Lambda}_f \boldsymbol{W}_{\boldsymbol{f},i}(1),\\
\frac{\boldsymbol{\tau}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} &\xrightarrow{T} W_i(1), \quad \frac{\boldsymbol{\Upsilon}_1'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} \xrightarrow{T} -2\pi\kappa\int_0^1 \cos(2\pi\kappa r)W_i(r)dr,\\
\frac{\boldsymbol{\Upsilon}_2'\boldsymbol{\varepsilon}_{iy}}{\sigma_i\sqrt{T}} &\xrightarrow{T} W(1) + 2\pi\kappa\int_0^1 \sin(2\pi\kappa r)W_i(r)dr,\\
\frac{\boldsymbol{\Upsilon}_1'\boldsymbol{s}_{f,-1}}{T^{3/2}} &\xrightarrow{T} -2\pi\kappa\left(\int_0^1 \cos(2\pi\kappa r)\boldsymbol{\Lambda}_f'\left[\int_0^r [\boldsymbol{W}_{\boldsymbol{f}}(s)]'ds\right]dr\right),\\
\frac{\boldsymbol{\Upsilon}_2'\boldsymbol{s}_{f,-1}}{T^{3/2}} &\xrightarrow{T} \boldsymbol{\Lambda}_f'\int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(s)]'ds + 2\pi\kappa\int_0^1 \sin(2\pi\kappa r)\boldsymbol{\Lambda}_f'\left[\int_0^r [\boldsymbol{W}_{\boldsymbol{f}}(s)]'ds\right]dr,\\
\frac{\boldsymbol{\Upsilon}_1'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} &\xrightarrow{T} -2\pi\kappa\left(\int_0^1 \cos(2\pi\kappa r)\left[\int_0^r W_i(s)ds\right]dr\right),\\
\frac{\boldsymbol{\Upsilon}_2'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} &\xrightarrow{T} \int_0^1 W_i(s)ds + 2\pi\kappa\int_0^1 \sin(2\pi\kappa r)\left[\int_0^r W_i(s)ds\right]dr,\\
\frac{\boldsymbol{s}_{f,-1}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i T} &\xrightarrow{T} \boldsymbol{\Lambda}_f\int_0^1 \boldsymbol{W}_{\boldsymbol{f}}(r)dW_i(r), \quad \frac{\boldsymbol{s}_{f,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i T^2} \xrightarrow{T} \boldsymbol{\Lambda}_f\int_0^1 \boldsymbol{W}_{\boldsymbol{f}}(r)W_i(r)dr,\\
\frac{\boldsymbol{s}_{f,-1}'\boldsymbol{s}_{f,-1}}{T^2} &\xrightarrow{T} \boldsymbol{\Lambda}_f\boldsymbol{\Lambda}_f'\int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(r)][\boldsymbol{W}_{\boldsymbol{f}}(r)]'dr, \quad \frac{\boldsymbol{F}'\boldsymbol{F}}{T} \xrightarrow{T} \boldsymbol{I}_m,\\
\frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} &\xrightarrow{T} \int_0^1 W_i^2(r)dr, \quad \frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} \xrightarrow{T} \int_0^1 W_i(r)dW_i(r) = \frac{1}{2}(W^2(1)-1)\\
\frac{\boldsymbol{F}'\boldsymbol{s}_{f,-1}}{\sigma_i T^{3/2}} &\xrightarrow{T} \boldsymbol{0}_{m\times m}, \quad \frac{\boldsymbol{\tau}'\boldsymbol{s}_{iy,-1}}{\sigma_i T^{3/2}} \xrightarrow{T} \int_0^1 W_i(r)dr, \quad \frac{\boldsymbol{F}'\boldsymbol{\tau}}{T} \xrightarrow{T} \boldsymbol{0}_{m\times 1}\\
\frac{\boldsymbol{F}'\boldsymbol{\Upsilon}_1}{T} &\xrightarrow{T} \boldsymbol{0}_{m\times 1}, \frac{\boldsymbol{F}'\boldsymbol{\Upsilon}_2}{T} \xrightarrow{T} \boldsymbol{0}_{m\times 1}, \quad \frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_1}{T} \xrightarrow{T} 0, \quad \frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2}{T} \xrightarrow{T} 0,
\end{aligned}
$$

where $W_i(r)$ is a scaler standard Brownian motion, and $\boldsymbol{W}_f(r)$ is a m-dimensional standard Brownian motion defined on [0,1] corresponding to ε_{iyt} and $\boldsymbol{v}_t$, respectively. Furthermore $W_i(r)$ and $\boldsymbol{W}_f(r)$ are mutually independent. By using the above results, terms in the numerator of the limiting distribution of the $t_i(N,T)$ statistic in (95) converge, when $T \to \infty$, as:

$$
\boldsymbol{q}_{iT} \xrightarrow{T} \boldsymbol{q}_{if}^*, \quad \boldsymbol{h}_{iT} \xrightarrow{T} \boldsymbol{h}_{if}^*, \quad \boldsymbol{\Psi}_{fT}^{-1} \xrightarrow{T} \boldsymbol{\Psi}_f^{*-1},
$$

where

$$\boldsymbol{q}_{if}^* = \begin{bmatrix} \boldsymbol{\Lambda}_f \boldsymbol{W}_{\boldsymbol{f},i}(1) \\ W_i(1) \\ -2\pi\kappa \int_0^1 \cos(2\pi\kappa r) W_i(r)dr \\ W(1) + 2\pi\kappa \int_0^1 \sin(2\pi\kappa r) W_i(r)dr \\ \boldsymbol{\Lambda}_f \int_0^1 \boldsymbol{W}_{\boldsymbol{f}}(r)dW_i(r) \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\Lambda}_f \boldsymbol{W}_{\boldsymbol{f},i}(1) \\ \boldsymbol{\Lambda}_f^* \boldsymbol{q}_{if} \end{bmatrix},$$

$$\boldsymbol{h}_{if}^* = \begin{bmatrix} \boldsymbol{0}_{m\times 1} \\ \int_0^1 W_i(r)dr \\ -2\pi\kappa \left(\int_0^1 \cos(2\pi\kappa r) \left[\int_0^r W_i(s)ds\right] dr\right) \\ \int_0^1 W_i(s)ds + 2\pi\kappa \int_0^1 \sin(2\pi\kappa r) \left[\int_0^r W_i(s)ds\right] dr \\ \boldsymbol{\Lambda}_f \int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(r)] W_i(r)dr \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{0}_{m\times 1} \\ \boldsymbol{\Lambda}_f^* \boldsymbol{h}_{if} \end{bmatrix},$$

$$\boldsymbol{\Psi}_f^* = \begin{bmatrix} \boldsymbol{I}_{m\times m} & \boldsymbol{0}'_{m\times(m+3)} \\ \boldsymbol{0}_{(m+3)\times m} & \boldsymbol{\Lambda}_f^* \boldsymbol{\Psi}_{f(m+3)\times(m+3)} \boldsymbol{\Lambda}_f^{*'} \end{bmatrix},$$

in which

$$\boldsymbol{\Lambda}_f^* = \begin{bmatrix} 1 & 0 & 0 & \boldsymbol{0} \\ 0 & 1 & 0 & \boldsymbol{0} \\ 0 & 0 & 1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Lambda}_f \end{bmatrix}, \text{ and } \boldsymbol{\Psi}_f = \begin{bmatrix} \boldsymbol{H}_{3\times 3} & \boldsymbol{R}_{3\times m} \\ \boldsymbol{R}'_{m\times 3} & \boldsymbol{J}_{m\times m} \end{bmatrix}$$

with

$$\boldsymbol{H}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \ \boldsymbol{R}_{3\times m} = \begin{bmatrix} \int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(r)]' dr \\ -2\pi\kappa \left(\int_0^1 \cos(2\pi\kappa r) \left[\int_0^r [\boldsymbol{W}_{\boldsymbol{f}}(s)]' ds\right] dr\right) \\ \int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(s)]' ds + 2\pi\kappa \int_0^1 \sin(2\pi\kappa r) \left[\int_0^r [\boldsymbol{W}_{\boldsymbol{f}}(s)]' ds\right] dr \end{bmatrix},$$

and $\boldsymbol{J}_{m\times m} = \int_0^1 [\boldsymbol{W}_{\boldsymbol{f}}(r)][\boldsymbol{W}_{\boldsymbol{f}}(r)]' dr$.

Since $\boldsymbol{\Lambda}_f^*$ is non-singular by Assumption 2 and $\boldsymbol{\Psi}_f$ is nonsingular, the numerator of the limiting distribution of the t-statistic in (95) is distributed, when $T \to \infty$, as:

$$\begin{aligned} \frac{\boldsymbol{\varepsilon}'_{iy} \boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \boldsymbol{q}'_{iT} \boldsymbol{\Psi}_{fT}^{-1} \boldsymbol{h}_{iT} \quad &\xrightarrow{T} \int_0^1 W_i(r)dW_i(r) - \boldsymbol{q}_{if}^{*'} \boldsymbol{\Psi}_f^{*-1} \boldsymbol{h}_{if}^*, \\ &= \int_0^1 W_i(r)dW_i(r) - \boldsymbol{q}'_{if} \boldsymbol{\Lambda}_f^{*'} (\boldsymbol{\Lambda}_f^* \boldsymbol{\Psi}_f \boldsymbol{\Lambda}_f^{*'})^{-1} \boldsymbol{\Lambda}_f^* \boldsymbol{h}_{if}, \\ &= \int_0^1 W_i(r)dW_i(r) - \boldsymbol{q}'_{if} \boldsymbol{\Psi}_f^{-1} \boldsymbol{h}_{if}. \end{aligned} \tag{96}$$

Similarly, the convergence results for the denominator in the right-hand side of (95) are:

$$\frac{\boldsymbol{\varepsilon}'_{iy} \boldsymbol{\varepsilon}_{iy}}{\sigma_i^2 (T-2k-6)} \xrightarrow{T} 1, \ \frac{\boldsymbol{d}'_{iT} \boldsymbol{\Xi}_{iT}^{-1} \boldsymbol{d}_{iT}}{T-2k-6} \xrightarrow{T} 0,$$

$$\frac{\boldsymbol{s}'_{iy,-1}\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} - \boldsymbol{h}'_{iT}\boldsymbol{\Psi}_{fT}^{-1}\boldsymbol{h}_{iT} \quad \xrightarrow{T} \quad \int_0^1 W_i^2(r)dr - \boldsymbol{h}'_{if}\boldsymbol{\Psi}_f^{-1}\boldsymbol{h}_{if}. \tag{97}$$

The joint convergence results in (96) and (97) together with the application of the continuous mapping theorem are sufficient to establish the stated result of sequential limit in Theorem 2.

It is easily seen that the asymptotic result from sequential limit also holds under joint limit so long as $\frac{N}{T} \to l$, where l is a fixed finite non-zero positive constant. The detailed proof is available from the authors upon request. This completes the proof of Theorem 2. ■

Proof of Theorem 3

Let $\boldsymbol{z}_{it}$ be generated by a unit root process with the Fourier form breaks defined as (9). The individual t-statistic in the panel unit-root test provided by Pesaran *et al.* (2013) is:

$$t_i^{PSY,B}(N,T) = \frac{\Delta\boldsymbol{y}'_i\bar{\boldsymbol{M}}\boldsymbol{y}_{i,-1}}{\left(\frac{\Delta\boldsymbol{y}'_i\bar{\boldsymbol{M}}_i\Delta\boldsymbol{y}_i}{T-2k-4}\right)^{1/2}\left(\boldsymbol{y}'_{i,-1}\bar{\boldsymbol{M}}\boldsymbol{y}_{i,-1}\right)^{1/2}}, \tag{98}$$

where $\bar{\boldsymbol{M}} = \boldsymbol{I}_T - \bar{\boldsymbol{W}}(\bar{\boldsymbol{W}}'\bar{\boldsymbol{W}})^{-1}\bar{\boldsymbol{W}}'$, $\bar{\boldsymbol{W}} = (\Delta\bar{\boldsymbol{z}}, \boldsymbol{\tau}, \bar{\boldsymbol{z}}_{-1})$, $\bar{\boldsymbol{M}}_i = \boldsymbol{I}_T - \bar{\boldsymbol{W}}_i(\bar{\boldsymbol{W}}'_i\bar{\boldsymbol{W}}_i)^{-1}\bar{\boldsymbol{W}}'_i$, $\bar{\boldsymbol{W}}_i = (\bar{\boldsymbol{W}}, \boldsymbol{y}_{i,-1})$. Note that Pesaran *et al.* (2013) does not contain the Fourier form breaks in DGP and hence the Fourier terms do not appear in their regression. The "residual maker" matrices, $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{M}}_i$ do not contain Fourier terms, $\boldsymbol{\Upsilon}_1$ and $\boldsymbol{\Upsilon}_2$.

Let the null process of Pesaran *et al.* (2013) be $\boldsymbol{y}_i^{PSY}$. Since Pesaran *et al.*'s (2013) model assumes away the Fourier breaks, the relation of the lag term and that of the first difference of the null process between the present paper and Pesaran *et al.* (2013) are:

$$\begin{aligned} \boldsymbol{y}_{i,-1} &= \boldsymbol{y}_{i,-1}^{PSY} + \alpha_{iy,1}\boldsymbol{\Upsilon}_{1,-1} + \alpha_{iy,2}\boldsymbol{\Upsilon}_{2,-1} \\ &= \boldsymbol{y}_{i,-1}^{PSY} + \alpha_{iy,1}(\boldsymbol{\Upsilon}_1 - \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2) + \alpha_{iy,2}(\boldsymbol{\Upsilon}_2 + \frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1) \\ &\equiv \boldsymbol{y}_{i,-1}^{PSY} + \breve{\alpha}_{iy,1}\boldsymbol{\Upsilon}_1 + \breve{\alpha}_{iy,2}\boldsymbol{\Upsilon}_2, \end{aligned} \tag{99}$$

$$\begin{aligned} \Delta\boldsymbol{y}_i &= \Delta\boldsymbol{y}_i^{PSY} + \alpha_{iy,1}\Delta\boldsymbol{\Upsilon}_1 + \alpha_{iy,2}\Delta\boldsymbol{\Upsilon}_2 \\ &= \Delta\boldsymbol{y}_i^{PSY} + \alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2 - \alpha_{iy,2}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1, \end{aligned} \tag{100}$$

where $\breve{\alpha}_{iy,1} = (\alpha_{iy,1} + \alpha_{iy,2}\frac{2\pi\kappa}{T})$, and $\breve{\alpha}_{iy,2} = (\alpha_{iy,2} - \alpha_{iy,1}\frac{2\pi\kappa}{T})$.

Pesaran *et al.* (2013, p.98) showed that $\bar{\boldsymbol{M}}\Delta\boldsymbol{y}_i^{PSY} = \sigma_i\bar{\boldsymbol{M}}\boldsymbol{v}_i$, $\bar{\boldsymbol{M}}\boldsymbol{y}_{i,-1}^{PSY} = \sigma_i\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}$, and $\bar{\boldsymbol{M}}_i\Delta\boldsymbol{y}_i^{PSY} = \sigma_i\bar{\boldsymbol{M}}_i\boldsymbol{v}_i$. By using (99) and (100), we can rewrite the $t_i^{PSY,B}(N,T)$ in (98) under the unit root hypothesis with the Fourier form breaks in the data generating process as:

$$t_i^{PSY,B}(N,T) \quad = \quad \frac{\frac{\sigma_i^2\boldsymbol{v}'_i\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T} + \lambda_3}{\left(\frac{\sigma_i^2\boldsymbol{v}'_i\bar{\boldsymbol{M}}_i\boldsymbol{v}_i}{T-2k-4} + \lambda_1\right)^{1/2} \times \left(\frac{\sigma_i^2\boldsymbol{\xi}'_{i,-1}\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T^2} + \lambda_2\right)^{1/2}}, \tag{101}$$

where

$$\lambda_1 = \frac{1}{T-2k-4}\left[2\sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{v}_i'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_2 - 2\sigma_i\alpha_{iy,2}\frac{2\pi\kappa}{T}\boldsymbol{v}_i'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_1 \right.$$
$$+\left(\alpha_{iy,1}\frac{2\pi\kappa}{T}\right)^2\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_2 + \left(\alpha_{iy,2}\frac{2\pi\kappa}{T}\right)^2\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_1$$
$$\left. - \alpha_{iy,1}\alpha_{iy,2}\left(\frac{2\pi\kappa}{T}\right)^2\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_1 - \alpha_{iy,1}\alpha_{iy,2}\left(\frac{2\pi\kappa}{T}\right)^2\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_2\right], \tag{102}$$

$$\lambda_2 = \frac{1}{T^2}\left[\sigma_i\breve{\alpha}_{iy,1}\boldsymbol{\xi}_{i,-1}'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1 + \sigma_i\breve{\alpha}_{iy,2}\boldsymbol{\xi}_{i,-1}'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2 + \sigma_i\breve{\alpha}_{iy,1}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}\right.$$
$$+\breve{\alpha}_{iy,1}\breve{\alpha}_{iy,1}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1 + \breve{\alpha}_{iy,1}\breve{\alpha}_{iy,2}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2 + \sigma_i\breve{\alpha}_{iy,2}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}$$
$$\left.+\breve{\alpha}_{iy,2}\breve{\alpha}_{iy,1}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1 + \breve{\alpha}_{iy,2}\breve{\alpha}_{iy,2}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2\right], \tag{103}$$

$$\lambda_3 = \frac{1}{T}\left[\alpha_{iy,1}\sigma_i\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1} - \alpha_{iy,2}\sigma_i\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1} + \sigma_i\breve{\alpha}_{iy,1}\boldsymbol{v}_i'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1\right.$$
$$+\alpha_{iy,1}\breve{\alpha}_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1 - \breve{\alpha}_{iy,1}\alpha_{iy,2}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_1 + \sigma_i\breve{\alpha}_{iy,2}\boldsymbol{v}_i'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2$$
$$\left.+\alpha_{iy,1}\breve{\alpha}_{iy,2}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2 - \alpha_{iy,2}\breve{\alpha}_{iy,2}\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1'\bar{\boldsymbol{M}}\boldsymbol{\Upsilon}_2\right]. \tag{104}$$

We now consider the limiting behavior of $t_i^{PSY,B}(N,T)$ as $N \to \infty$. To calculate the order of probability for each element in λ_i for $i = 1, 2, 3$ as $N \to \infty$, we first collect the results that have been proved or assumed. They are: (a) $\boldsymbol{\alpha}_1^* = \boldsymbol{O}(1)_{(k+1)\times 1}$, $\boldsymbol{\alpha}_2^* = \boldsymbol{O}(1)_{(k+1)\times 1}$, and $\boldsymbol{\Gamma}^* = \boldsymbol{O}(1)_{(k+1)\times m}$. (by Assumptions 3 and 5), (b) $\boldsymbol{v}_i'\bar{\boldsymbol{W}}\boldsymbol{B}_1 = \boldsymbol{O}(1)_{1\times(2k+3)}$, and $\boldsymbol{v}_i'\bar{\boldsymbol{W}}_i\boldsymbol{B}_2 = \boldsymbol{O}(1)_{1\times(2k+4)}$. (Pesaran *et al.*, 2009, p.23), and (c) $\boldsymbol{B}_1\bar{\boldsymbol{W}}'\bar{\boldsymbol{W}}\boldsymbol{B}_1 = \boldsymbol{O}(1)_{(2k+3)\times(2k+3)}$, and $\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2 = \boldsymbol{O}(1)_{(2k+4)\times(2k+4)}$, where $\boldsymbol{B}_1 = diag(T^{-1/2}\boldsymbol{I}_{k+2},\ T^{-1}\boldsymbol{I}_{k+1})$ and $\boldsymbol{B}_2 = diag(\boldsymbol{B}_1,\ T^{-1})$ are defined in Pesaran *et al.* (2013, pp.106-107).

In view of the above results, the order of the probability for the first term in λ_1 can be calculated as:

$$\frac{\sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{v}_i'\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_2}{T-2k-4} = \frac{\sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{v}_i'\boldsymbol{\Upsilon}_2}{T-2k-4} - \sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}(\boldsymbol{v}_i'\bar{\boldsymbol{W}}_i\boldsymbol{B}_2)(\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2)^{-1} \times\left(\frac{\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\boldsymbol{\Upsilon}_2}{T-2k-4}\right). \tag{105}$$

To investigate the order of the probability for each element in (105), we first consider the first element in the right-hand side of (105). According to the definition of $\boldsymbol{v}_i$ in (43), as $N \to \infty$,

$$\frac{\sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{v}_i'\boldsymbol{\Upsilon}_2}{T-2k-4} \overset{N}{\longrightarrow} 2\pi\kappa\alpha_{iy,1}\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\Upsilon}_2}{T(T-2k-4)} = O(1)O(T^{-3/2}) = O(T^{-3/2}). \tag{106}$$

This is because $\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\Upsilon}_2}{T^{1/2}} = O(1)$ from Lemma (L7). We next consider the order of probability for the last term in the right-hand side of (105), $\frac{\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\boldsymbol{\Upsilon}_2}{T-2k-4}$, and note that

$$\frac{\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\boldsymbol{\Upsilon}_2}{T-2k-4}=\begin{pmatrix}\frac{\Delta\bar{\boldsymbol{z}}'}{T^{1/2}(T-2k-4)}\\ \frac{\boldsymbol{\tau}'}{T^{1/2}(T-2k-4)}\\ \frac{\bar{\boldsymbol{z}}'_{-1}}{T(T-2k-4)}\\ \frac{\boldsymbol{y}'_{i,-1}}{T(T-2k-4)}\end{pmatrix}\boldsymbol{\Upsilon}_2\xrightarrow{N}\begin{pmatrix}\frac{(\boldsymbol{\Gamma}^*\boldsymbol{F}'+\bar{\boldsymbol{A}}^*\Delta\boldsymbol{D}')\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)}\\ \frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)}\\ \frac{(\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\bar{\boldsymbol{A}}^*\boldsymbol{D}'_{-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4}\\ \frac{(y_{i0}\boldsymbol{\tau}'+\boldsymbol{\alpha}'_{iy}\boldsymbol{D}'_{-1}+\boldsymbol{\delta}'_i\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\boldsymbol{s}'_{iy,-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4)}\end{pmatrix}. \qquad (107)$$

The first element in the vector of (107) is (see equation (85)):

$$\begin{aligned}\frac{(\boldsymbol{\Gamma}^*\boldsymbol{F}'+\bar{\boldsymbol{A}}^*\Delta\boldsymbol{D}')\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)} &= \frac{\boldsymbol{\Gamma}^*\boldsymbol{F}'\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)}+\frac{\left(\frac{2\pi\kappa}{T}\boldsymbol{\alpha}_1^*\boldsymbol{\Upsilon}'_2-\frac{2\pi\kappa}{T}\boldsymbol{\alpha}_2^*\boldsymbol{\Upsilon}'_1\right)\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)},\\ &= \boldsymbol{\Gamma}^*\frac{\boldsymbol{F}'\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)}+2\pi\kappa\frac{\boldsymbol{\alpha}_1^*\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2-\boldsymbol{\alpha}_2^*\boldsymbol{\Upsilon}'_1\boldsymbol{\Upsilon}_2}{T^{3/2}(T-2k-4)}.\end{aligned}$$

Because $2\pi\kappa=O(1)$, $\boldsymbol{\Gamma}^*=\boldsymbol{O}(1)_{(k+1)\times m}$, $\frac{\boldsymbol{F}'\boldsymbol{\Upsilon}_2}{\sqrt{T}}=\boldsymbol{O}(1)_{m\times 1}$, $\boldsymbol{\alpha}_1^*=\boldsymbol{O}(1)_{(k+1)\times 1}$, $\frac{\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2}{T}=O(1)$, and $\boldsymbol{\Upsilon}'_1\boldsymbol{\Upsilon}_2=0$ (By Lemmas (L2) and (L3)), thus, as $N\to\infty$,

$$\begin{aligned}\frac{(\boldsymbol{\Gamma}\boldsymbol{F}'+\bar{\boldsymbol{A}}^*\Delta\boldsymbol{D}')\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)} &= \left[\boldsymbol{O}(1)_{(k+1)\times m}\boldsymbol{O}(T^{-1})_{m\times 1}\right]+\left[O(1)\boldsymbol{O}(1)_{(k+1)\times 1}O(T^{-3/2})\right],\\ &\equiv \boldsymbol{O}(T^{-1})_{(k+1)\times 1}. \qquad (108)\end{aligned}$$

The second element in the vector of (107) is:

$$\frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2}{T^{1/2}(T-2k-4)} = O(T^{-3/2}). \qquad (109)$$

This is because $\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2=O(1)$ by Lemma (L4). The third element in the vector of (107) is

$$\frac{(\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\bar{\boldsymbol{A}}^*\boldsymbol{D}'_{-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4)}=\boldsymbol{\Gamma}^*\frac{\boldsymbol{s}'_{f,-1}\boldsymbol{\Upsilon}_2}{T(T-2k-4)}+\boldsymbol{\alpha}_1^{**}\frac{\boldsymbol{\Upsilon}'_1\boldsymbol{\Upsilon}_2}{T(T-2k-4)}+\boldsymbol{\alpha}_2^{**}\frac{\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2}{T(T-2k-4)}.$$

Since $\frac{\boldsymbol{s}'_{f,-1}\boldsymbol{\Upsilon}_2}{T^{3/2}}=\boldsymbol{O}(1)_{m\times 1}$ (from Lemma (L10)), $\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_1=0$, $\frac{\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2}{T}=O(1)$, $\boldsymbol{\alpha}_1^{**}=\boldsymbol{\alpha}_1^*+\boldsymbol{\alpha}_2^*\frac{2\pi\kappa}{T}=\boldsymbol{O}(1)_{(k+1)\times 1}+\boldsymbol{O}(T^{-1})_{(k+1)\times 1}$, and $\boldsymbol{\alpha}_2^{**}=\boldsymbol{\alpha}_2^*-\boldsymbol{\alpha}_1^*\frac{2\pi\kappa}{T}=\boldsymbol{O}(1)_{(k+1)\times 1}+\boldsymbol{O}(T^{-1})_{(k+1)\times 1}$ (by Assumption 3), thus, as $N\to\infty$,

$$\begin{aligned}\frac{(\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\bar{\boldsymbol{A}}^*\boldsymbol{D}'_{-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4)} &= \boldsymbol{O}(1)_{(k+1)\times m}\boldsymbol{O}(T^{-1/2})_{m\times 1}+\boldsymbol{O}(1)_{(k+1)\times 1}O(T^{-2})\\ &\quad +\boldsymbol{O}(1)_{(k+1)\times 1}O(T^{-1})\equiv\boldsymbol{O}(T^{-1/2})_{(k+1)\times 1}. \qquad (110)\end{aligned}$$

The fourth element in the vector of (107):

$$\begin{aligned}&\frac{(y_{i0}\boldsymbol{\tau}'+\boldsymbol{\alpha}'_{iy}\boldsymbol{D}'_{-1}+\boldsymbol{\delta}'_i\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\boldsymbol{s}'_{iy,-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4)}\\ &= y_{i0}\frac{\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2}{T(T-2k-4)}+\boldsymbol{\alpha}'_{iy}\frac{\boldsymbol{D}'_{-1}\boldsymbol{\Upsilon}_2}{T(T-2k-4)}+\boldsymbol{\delta}'_i\boldsymbol{\Gamma}^*\frac{\boldsymbol{s}'_{f,-1}\boldsymbol{\Upsilon}_2}{T(T-2k-4)}+\frac{\boldsymbol{s}'_{iy,-1}\boldsymbol{\Upsilon}_2}{T(T-2k-4)},\end{aligned}$$

in which

$$\frac{\boldsymbol{D}'_{-1}\boldsymbol{\Upsilon}_2}{T(T-2k-4)} = \frac{[\boldsymbol{\Upsilon}_{1,-1},\boldsymbol{\Upsilon}_{2,-1}]'\boldsymbol{\Upsilon}_2}{T(T-2k-4)} = \frac{[\boldsymbol{\Upsilon}_1-\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}_2+\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}_1]'\boldsymbol{\Upsilon}_2}{T(T-2k-4)}$$

$$= \frac{[\boldsymbol{\Upsilon}'_1\boldsymbol{\Upsilon}_2-\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2,\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2+\frac{2\pi\kappa}{T}\boldsymbol{\Upsilon}'_1\boldsymbol{\Upsilon}_2]}{T(T-2k-4)}.$$

Since $y_{i0}=O(1)$ (by Assumption 4), $\boldsymbol{\tau}'\boldsymbol{\Upsilon}_2=O(1)$ (from Lemma (L4)), $\boldsymbol{\alpha}'_{iy}=\boldsymbol{O}(1)_{1\times 2}$ (by Assumption 3), $\frac{\boldsymbol{s}'_{f,-1}\boldsymbol{\Upsilon}_2}{T^{3/2}}=\boldsymbol{O}(1)_{m\times 1}$ (from Lemma (L10)), $\bar{\boldsymbol{\Gamma}}^*=\boldsymbol{O}(1)_{(k+1)\times m}$, $\boldsymbol{\delta}_i=\boldsymbol{\Gamma}^*(\boldsymbol{\Gamma}^{*'}\boldsymbol{\Gamma}^*)^{-1}\boldsymbol{\gamma}_{iy}=\boldsymbol{O}(1)_{(k+1)\times 1}$ (by Assumptions 3 and 5), $\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_1=0$, $\frac{\boldsymbol{\Upsilon}'_2\boldsymbol{\Upsilon}_2}{T}=O(1)$, and $\frac{\boldsymbol{s}'_{iy,-1}\boldsymbol{\Upsilon}_2}{T^{3/2}}=O(1)$, thus, as $N\to\infty$, the fourth element in the vector of (107):

$$\begin{aligned}
&\frac{(y_{i0}\boldsymbol{\tau}'+\boldsymbol{\alpha}'_{iy}\boldsymbol{D}'_{-1}+\boldsymbol{\delta}'_i\boldsymbol{\Gamma}^*\boldsymbol{s}'_{f,-1}+\boldsymbol{s}'_{iy,-1})\boldsymbol{\Upsilon}_2}{T(T-2k-4)}\\
=\;& O(1)O(T^{-2})+\boldsymbol{O}(1)_{1\times 2}\cdot[O(T^{-2}),O(T^{-1})]+\boldsymbol{O}(1)_{1\times(k+1)}\boldsymbol{O}(1)_{(k+1)\times m}\boldsymbol{O}(T^{-1/2})_{m\times 1}\\
&+O(T^{-1/2})\\
\equiv\;& O(T^{-1/2}). \qquad (111)
\end{aligned}$$

By substituting (108)-(111) into (107) and using the results $\boldsymbol{v}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2=\boldsymbol{O}(1)_{1\times(2k+4)}$ and $\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2=\boldsymbol{O}(1)_{(2k+4)\times(2k+4)}$, we conclude that the order of the probability for the second element in the right-hand side of (105) is:

$$\begin{aligned}
&\alpha_{iy,1}\cdot\frac{2\pi\kappa}{T}\sigma_i(\boldsymbol{v}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2)(\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\bar{\boldsymbol{W}}_i\boldsymbol{B}_2)^{-1}\left(\frac{\boldsymbol{B}_2\bar{\boldsymbol{W}}'_i\boldsymbol{\Upsilon}_2}{T-2k-4}\right)\\
&=O(T^{-1})\cdot\left\{\left[\boldsymbol{O}(1)_{1\times(2k+4)}\cdot\boldsymbol{O}(1)_{(2k+4)\times(2k+4)}\right]\cdot\begin{pmatrix}\boldsymbol{O}(T^{-1})_{(k+1)\times 1}\\ O(T^{-3/2})_{1\times 1}\\ \boldsymbol{O}(T^{-1/2})_{(k+1)\times 1}\\ O(T^{-1/2})_{1\times 1}\end{pmatrix}\right\},\\
&\equiv O(T^{-3/2}). \qquad (112)
\end{aligned}$$

By substituting (106) and (112) into (105), the order of the probability for the first term in λ_1, $\frac{\sigma_i\alpha_{iy,1}\frac{2\pi\kappa}{T}\boldsymbol{v}'_i\bar{\boldsymbol{M}}_i\boldsymbol{\Upsilon}_2}{T-2k-4}$, is $O(T^{-3/2})$. Similarly, it can be shown that the order of probability is $O(T^{-3/2})$ for the second term and is $O(T^{-2})$ for the third, fourth, fifth and sixth terms in λ_1. We therefore conclude that $\lambda_1=O(T^{-3/2})$ as $N\to\infty$.

It can also be shown that $\lambda_2=O(T^{-1/2})$ and $\lambda_3=O(T^{-1/2})$ after the same straightforward but tedious algebra.[23] Pesaran *et al.* (2013) showed that:

$$\frac{\frac{\sigma_i^2\boldsymbol{v}'_i\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T}}{\left(\frac{\sigma_i^2\boldsymbol{v}'_i\bar{\boldsymbol{M}}_i\boldsymbol{v}_i}{T-2k-4}\times\frac{\sigma_i^2\boldsymbol{\xi}'_{i,-1}\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T^2}\right)^{1/2}}\xrightarrow{N}\frac{\frac{\boldsymbol{\varepsilon}'_{iy}\boldsymbol{s}_{iy,-1}}{\sigma_i^2T}-\mathring{\boldsymbol{q}}'_{iT}\boldsymbol{\Upsilon}^{-1}_{fT}\mathring{\boldsymbol{h}}_{iT}}{J_1^{p*}\times J_2^{p*}}=O(1), \qquad (113)$$

[23]The detailed proof is available from the authors upon request.

where $J_1^{p*} = \left(\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{\varepsilon}_{iy}}{\sigma_i^2(T-2k-4)} - \frac{\boldsymbol{g}_{iT}'\boldsymbol{Q}_{iT}^{-1}\boldsymbol{g}_{iT}}{T-2k-4}\right)^{1/2} = O(1)$ and $J_2^{p*} = \left(\frac{\boldsymbol{s}_{iy,-1}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T^2} - \mathring{\boldsymbol{h}}_{iT}'\boldsymbol{\Upsilon}_{fT}^{-1}\mathring{\boldsymbol{h}}_{iT}\right)^{1/2} = O(1)$, in which $\boldsymbol{g}_{iT}$, $\boldsymbol{Q}_{iT}$, $\mathring{\boldsymbol{q}}_{iT}$, $\boldsymbol{\Upsilon}_{fT}$, and $\mathring{\boldsymbol{h}}_{iT}$ are defined in (26). By combining the results in (113) and those of the order of the probability for λ_1, λ_2 and λ_3, we obtain:

$$
\begin{aligned}
t_i^{PSY,B} &= \frac{\frac{\sigma_i^2\boldsymbol{v}_i'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T} + \lambda_3}{\left(\frac{\sigma_i^2\boldsymbol{v}_i'\bar{\boldsymbol{M}}_i\boldsymbol{v}_i}{T-2k-4} + \lambda_1\right)^{1/2} \times \left(\frac{\sigma_i^2\boldsymbol{\xi}_{i,-1}'\bar{\boldsymbol{M}}\boldsymbol{\xi}_{i,-1}}{T^2} + \lambda_2\right)^{1/2}} \\
&\xrightarrow{N} \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \mathring{\boldsymbol{q}}_{iT}'\boldsymbol{\Upsilon}_{fT}^{-1}\mathring{\boldsymbol{h}}_{iT} + O(T^{-1/2})}{J_1^{p*} \times J_2^{p*} + [O(T^{-3/2})]^{1/2}J_2^{p*}\sigma_i + [O(T^{-1/2})]^{1/2}J_1^{p*}\sigma_i + [O(T^{-1/2})O(T^{-3/2})]^{1/2}} \\
&= \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \mathring{\boldsymbol{q}}_{iT}'\boldsymbol{\Upsilon}_{fT}^{-1}\mathring{\boldsymbol{h}}_{iT} + O(T^{-1/2})}{J_1^{p*} \times J_2^{p*} + [O(T^{-3/2})]^{1/2}J_2^{p*}\sigma_i + [O(T^{-1/2})]^{1/2}J_1^{p*}\sigma_i + [O(T^{-2})]^{1/2}} \\
&\equiv \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \mathring{\boldsymbol{q}}_{iT}'\boldsymbol{\Upsilon}_{fT}^{-1}\mathring{\boldsymbol{h}}_{iT} + O(T^{-1/2})}{J_1^{p*} \times J_2^{p*} + O(T^{-1/4})} \\
&= \frac{\frac{\boldsymbol{\varepsilon}_{iy}'\boldsymbol{s}_{iy,-1}}{\sigma_i^2 T} - \mathring{\boldsymbol{q}}_{iT}'\boldsymbol{\Upsilon}_{fT}^{-1}\mathring{\boldsymbol{h}}_{iT}}{J_1^{p*} \times J_2^{p*}} \oplus \frac{O(T^{-1/2})}{O(T^{-1/4})},
\end{aligned} \tag{114}
$$

where the notation "$\oplus$" is adapted from the Farey sequence to denote $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$.

If next to N, T also tends to infinity then, as in Pesaran *et al.* (2013), we have:

$$
t_i^{PSY,B} \xrightarrow{(N,T)_{seq}} \frac{\int_0^1 W_i(r)dW_i(r) - \boldsymbol{\omega}_{i\boldsymbol{v}}'\boldsymbol{G}_{\boldsymbol{v}}^{-1}\boldsymbol{\pi}_{i\boldsymbol{v}}}{\left(\int_0^1 W_i^2(r)dr - \boldsymbol{\pi}_{i\boldsymbol{v}}'\boldsymbol{G}_{\boldsymbol{v}}^{-1}\boldsymbol{\pi}_{i\boldsymbol{v}}\right)^{1/2}}.
$$

This completes the proof of Theorem 3. ■

Proof of Theorem 4

Following the procedures of proving Theorems 1, 2 and the Theorem 2.3 in Pesaran *et al.* (2009), it is straightforward to prove Theorem 4. The detailed proof is available from the authors upon request. ■

The sketch of the proof for Remark 1

Under the assumption of heterogeneous frequencies across individuals, i.e. $\kappa_i \neq \kappa_j, \forall i \neq j,\ i,j \in N$, it is straightforward to show that $t_i(N,T,\kappa_i)$, with serially uncorrelated errors, has the following limiting distribution (replacing κ with κ_i in Theorem 2):

$$
t_i(N,T,\kappa_i) \xrightarrow{(N,T)_{seq}} BCADF_{if,\kappa_i} \equiv \frac{\int_0^1 W_i(r)dW_i(r) - \boldsymbol{q}_{if,\kappa_i}'\boldsymbol{\Psi}_{f,\kappa_i}^{-1}\boldsymbol{h}_{if,\kappa_i}}{\left(\int_0^1 W_i^2(r)d(r) - \boldsymbol{h}_{if,\kappa_i}'\boldsymbol{\Psi}_{f,\kappa_i}^{-1}\boldsymbol{h}_{if,\kappa_i}\right)^{1/2}},
$$

where

$$
\begin{aligned}
\boldsymbol{q}_{if,\kappa_i} &= \begin{bmatrix} W_i(1) \\ -2\pi\kappa_i \int_0^1 \cos(2\pi\kappa_i r) W_i(r)dr \\ W(1) + 2\pi\kappa_i \int_0^1 \sin(2\pi\kappa_i r) W_i(r)dr \\ \int_0^1 \boldsymbol{W_f}(r) dW_i(r) \end{bmatrix}, \\
\boldsymbol{h}_{if,\kappa_i} &= \begin{bmatrix} \int_0^1 W_i(r)dr \\ -2\pi\kappa_i \left(\int_0^1 \cos(2\pi\kappa_i r) \left[\int_0^r W_i(s)ds\right] dr \right) \\ \int_0^1 W_i(s)ds + 2\pi\kappa_i \int_0^1 \sin(2\pi\kappa_i r) \left[\int_0^r W_i(s)ds\right] dr \\ \int_0^1 [\boldsymbol{W_f}(r)] W_i(r)dr \end{bmatrix}, \\
\boldsymbol{\Psi}_{f,\kappa_i} &= \begin{bmatrix} \boldsymbol{H}_{3\times 3} & \boldsymbol{R}_{\kappa_i,3\times m} \\ \boldsymbol{R}'_{\kappa_i,m\times 3} & \boldsymbol{J}_{m\times m} \end{bmatrix}, \text{ with } \boldsymbol{H}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, \\
\boldsymbol{R}_{\kappa_i,3\times m} &= \begin{bmatrix} \int_0^1 [\boldsymbol{W_f}(r)]' dr \\ -2\pi\kappa_i \left(\int_0^1 \cos(2\pi\kappa_i r) \left[\int_0^r [\boldsymbol{W_f}(s)]' ds\right] dr \right) \\ \int_0^1 [\boldsymbol{W_f}(s)]' ds + 2\pi\kappa_i \int_0^1 \sin(2\pi\kappa_i r) \left[\int_0^r [\boldsymbol{W_f}(s)]' ds\right] dr \end{bmatrix},
\end{aligned}
$$

and $\boldsymbol{J}_{m\times m} = \int_0^1 [\boldsymbol{W_f}(r)][\boldsymbol{W_f}(r)]' dr$. $W_i(r)$ and $\boldsymbol{W_f}(r)$ are scalar and m-dimensional standard Brownian motions, respectively, and they are mutually independent. Again, the limiting distribution of $t_i(N, T, \kappa_i)$, $\forall i$, depends on the common process $\boldsymbol{W_f}(r)$. ■

The sketch of the procedure proposed in Remark 2

Following Bai and Ng (2004), the common effects from factors can be removed by using the de-factor method. Hence, the standardized t-bar statistic in IPS (2003) and ILT (2010) can be applied.

The data generating process of y_{it} in our model with more general assumptions is:

$$(1 - \phi_i L)(y_{it} - \ddot{\delta}_i(t) - \varsigma_i t) = \boldsymbol{\gamma}'_{iy} \boldsymbol{f}_t + \ddot{\eta}_{iyt},$$

which can be rewritten as:

$$y_{it} = \varsigma_i t + \ddot{\delta}_i(t) + \boldsymbol{\gamma}'_{iy} \ddot{\boldsymbol{f}}_t + \ddot{e}_{it}, \tag{115}$$

where

$$
\begin{aligned}
\ddot{\delta}_i(t) &= \mu_i + \alpha_{iy,1} \sin(2\pi\kappa_i t/T) + \alpha_{iy,2} \cos(2\pi\kappa_i t/T), \\
(1 - \phi_i L)\ddot{\boldsymbol{f}}_t &= \boldsymbol{f}_t, \\
(1 - \phi_i L)\ddot{e}_{it} &= \ddot{\eta}_{iyt} = D_i(L)\varepsilon_{iyt},
\end{aligned}
$$

with $D_i(L) = \sum_{j=0}^{\infty} D_{ij} L^j = (1 - \rho_{i,1} L - \cdots - \rho_{i,l_i} L^{l_i})^{-1}(1 + \theta_{i,1} L + \cdots + \theta_{i,s_i} L^{s_i})$. Here we do not impose the homogeneity assumptions on Fourier frequencies ($\kappa_i \neq \kappa$) and lag orders ($l_i \neq l,\ s_i \neq s$). This is a special case of the Bai and Ng (2004) model where y_{it} is $I(1)$ when both $\ddot{\boldsymbol{f}}_t$ and $\ddot{e}_{it}$ are $I(1)$ simultaneously.

Taking the first difference of (115), we obtain:

$$\Delta y_{it} = \varsigma_i + \alpha_{iy,1}\Delta\sin(2\pi\kappa_i t/T) + \alpha_{iy,2}\Delta\cos(2\pi\kappa_i t/T) + \boldsymbol{\gamma}'_{iy}\Delta\ddot{\boldsymbol{f}}_t + \Delta\ddot{e}_{it}.$$

Applying the method of principal components to Δy_{it} (after demeaned and de-Fourier terms), we obtain m estimated factors $\widehat{\Delta\ddot{\boldsymbol{f}}}_t$, the associated loadings $\hat{\boldsymbol{\gamma}}_{iy}$, and the estimated residuals $\omega_{it} \equiv \widehat{\Delta\ddot{e}}_{it} = \Delta y_{it} - \hat{\varsigma}_i - \hat{\alpha}_{iy,1}\Delta\sin(2\pi\kappa_i t/T) - \hat{\alpha}_{iy,2}\Delta\cos(2\pi\kappa_i t/T) - \hat{\boldsymbol{\gamma}}'_{iy}\widehat{\Delta\ddot{\boldsymbol{f}}}_t$. Define for $t = 2, ..., T$:

$$\hat{\ddot{e}}_{it} = \sum_{t=2}^{T}\omega_{it}.$$

Let $ADF^f_{\hat{\ddot{e}}}(i)$ be the t-statistic for testing $d_{io} = 0$ in the univariate augmented autoregression:

$$\Delta\hat{\ddot{e}}_{it} = d_{io}\hat{\ddot{e}}_{it} + d_{i1}\Delta\hat{\ddot{e}}_{it-1} + \cdots + d_{ip_i}\Delta\hat{\ddot{e}}_{it-p_i} + \text{error},$$

where p_i is the selected order of autoregression that satisfies certain deterministic rate conditions (for example, $p_i \to \infty$ and $p_i^3/\min[N,T] \to 0$ in Bai and Ng (2004)). The limiting distribution of $ADF^f_{\hat{\ddot{e}}}(i)$ under the null hypothesis $\phi_i = 1$ is expected to be the function of Brownian motions driven by ε_{iyt} with parameter κ_i.

The univariate tests for $\hat{\ddot{e}}_{it}$ do not depend on Brownian motions driven by the common factor f_t asymptotically, i.e., they are cross-sectionally independent. The panel unit root test statistic can be obtained as the standardized statistic of the following average test statistic:

$$\bar{t} = \frac{1}{N}\sum_{i=1}^{N} ADF^f_{\hat{\ddot{e}}}(i),$$

i.e.,

$$\overline{ADF^f_{\hat{\ddot{e}}}} = \frac{\sqrt{N}\left[\bar{t} - \tilde{E}(\bar{t})\right]}{\sqrt{\tilde{V}(\bar{t})}},$$

where $\tilde{E}(\bar{t})$ and $\tilde{V}(\bar{t})$ are the estimates of the mean and variance of $\bar{t}$, $E(\bar{t})$ and $V(\bar{t})$, which are functions of κ_i and p_i. The panel test statistic $\overline{ADF^f_{\hat{\ddot{e}}}}$ would follow a standard normal distribution under the null hypothesis. ■